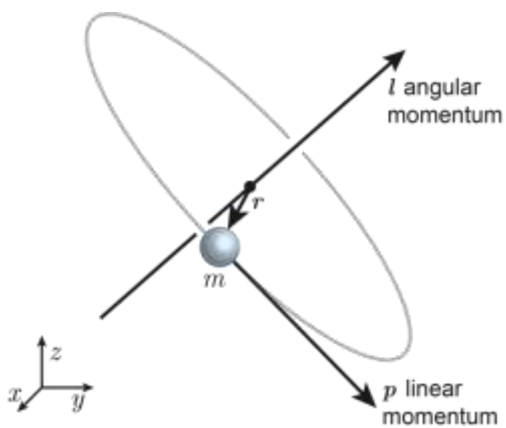
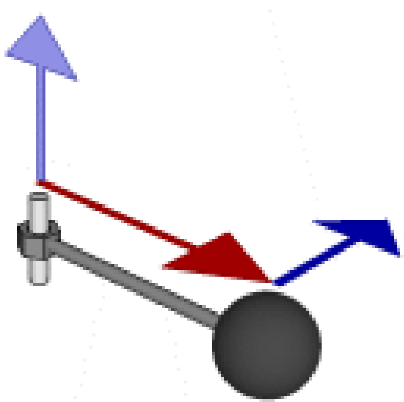
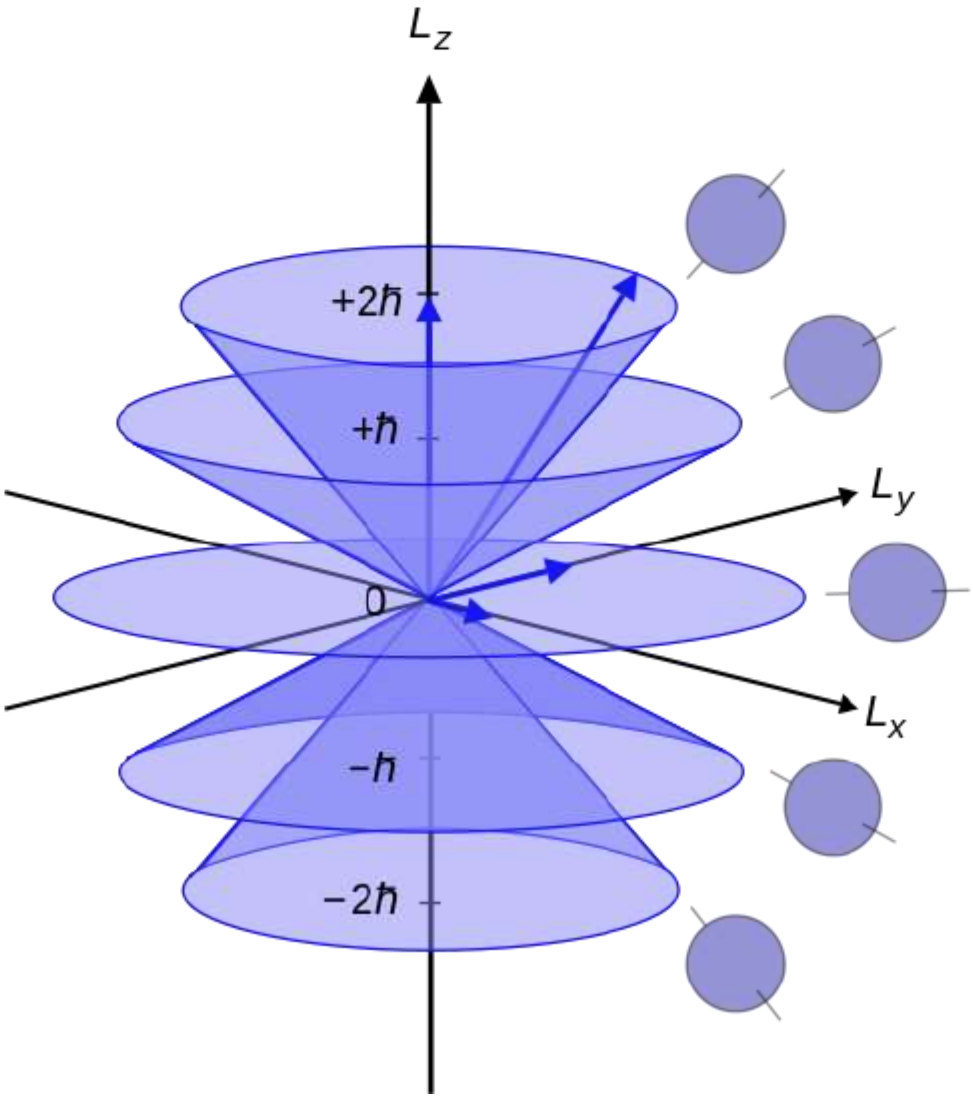
Angular Momentum

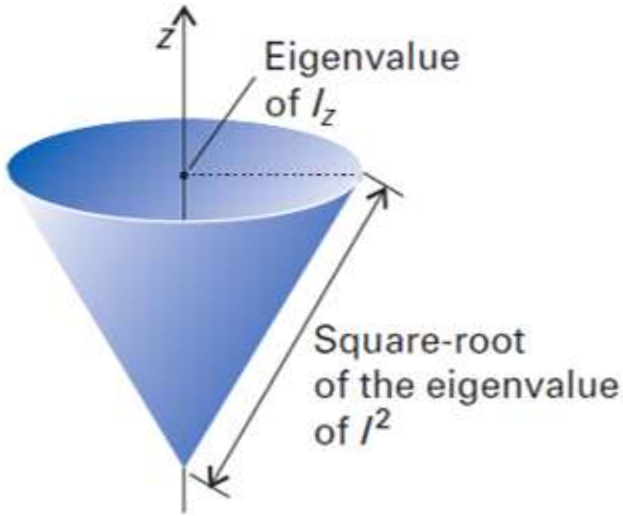


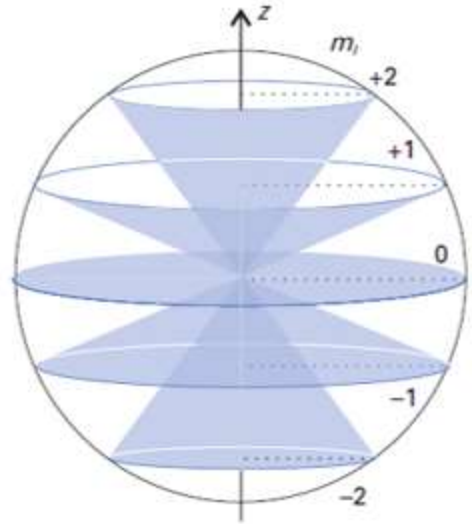


$\tau = \mathbf{r} \times \mathbf{F}$

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

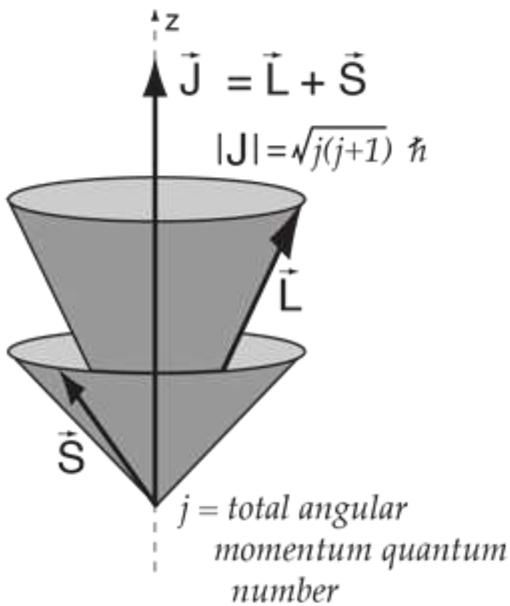


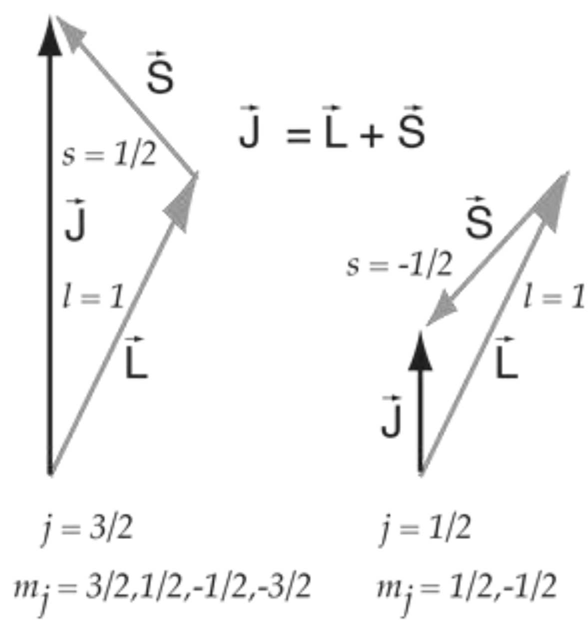




Projection of Azimuthal and angular momentum

Space quantization





Spin-orbit coupling

Angular Momentum:-

⇒ There are electrons in an atom. Since the electrons move in an angular path, 'Angular momentum' becomes one of the most important factor for chemists to describe the atomic states.

⇒ Angular Momentum is of three types.

(1) Spin Angular Momentum :- Electron, the microscopic particles possess some intrinsic/inherent angular momentum i.e around their own axis. Denoted by S.

(2) Orbital Angular Momentum :- It arises due to the movement of electron in the orbital. It's Denoted by 'L'.

(3) Total angular Momentum :- Due to the vector coupling of L & S.

⇒ From the classical mechanics, momentum can be defined as the moment of linear momentum

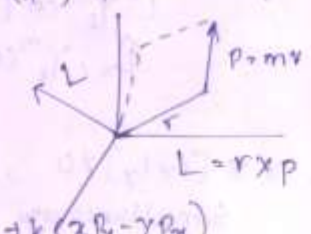
i.e Angular Momentum = $r \times mv = mvr$.

L is given by, $L = iL_x + jL_y + kL_z$ — ①

where, L_x, L_y & L_z are the different projection along the three axes and i, j, k are unit vectors.

Then $\vec{L} = |r||p|$, where $\vec{r} = ix + jy + kz$, $\vec{p} = iP_x + jP_y + kP_z$ — (ii)

$$\Rightarrow L \begin{vmatrix} i & j & k \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$



$$\therefore L = i(yP_z - zP_y) - j(xP_z - zP_x) + k(xP_y - yP_x)$$

$$= i(yP_z - zP_y) + j(zP_x + xP_z) + k(xP_y - yP_x) \quad \text{— (iii)}$$

From quantum mechanics P_x, P_y and P_z are the components of linear momentum, which are three observable corresponding to each of these, there exist an operator (according to 1st postulate of quantum mechanics).

$$\therefore \hat{P}_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \quad \hat{P}_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \quad \& \quad \hat{P}_z = \frac{\hbar}{i}\frac{\partial}{\partial z} \quad \text{— (iv)}$$

x, y & z are simple positional operators

$$\text{Now } L = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right) + \left(z\frac{\hbar}{i}\frac{\partial}{\partial x} - x\frac{\hbar}{i}\frac{\partial}{\partial z}\right) + x\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \quad \text{— (v)}$$

$$= \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\Rightarrow L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = \hbar/i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$\Rightarrow$ L^2 - orbital angular momentum squared

It gives the idea about 'l' (quantum no)

L^2 and $\hat{L}_z$ commute, According to quantum mechanics theorem if two operator commute then they have same set of eigen function of L^2 is $Y_l^m(\theta, \phi)$ → spherical harmonics.

The eigen value equation in atomic unit is

$$L^2 Y_l^m(\theta,\phi) = l(l+1) Y_l^m(\theta,\phi) \qquad [H, L^2] = 0$$

$$\hat{L}_z, Y_l^m(\theta,\phi) = m Y_l^m(\theta,\phi) \qquad [L^2, L_i] = 0$$

** $[L_x L_y] \neq 0$, $[L_x, L_y]$ don't commute i.e the simultaneous measurement of the two observable is impossible.

⇒ That is the importance of uncertainity Principle.

Q. Find out $[L_x, L_y]$

Ans:- $[L_x, L_y] = L_xL_y - L_yL_x$

$$= \left[\frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]\left[\frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] - \left[\frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]\left[\frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$= \frac{\hbar^2}{i^2}\left(y\frac{\partial}{\partial z}\cdot z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\cdot x\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\, z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\cdot x\frac{\partial}{\partial z}\right) - \frac{\hbar^2}{i^2}\left(z\frac{\partial}{\partial x}\cdot y\frac{\partial}{\partial z}\right.$$

$$\left. - x\frac{\partial}{\partial z}\cdot y\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\cdot z\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\cdot z\frac{\partial}{\partial y}\right)$$

We can write $y\frac{\partial}{\partial z}\cdot z\frac{\partial}{\partial x}$ as $\left(y\frac{\partial z}{\partial z}\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial x}\right) = y\cdot 1\cdot\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x}$

Similarly, $x\frac{\partial}{\partial z}\cdot z\frac{\partial}{\partial y} = \left(x\frac{\partial z}{\partial z}\frac{\partial}{\partial y} + xz\frac{\partial^2}{\partial z\partial y}\right) = x\cdot 1\cdot\frac{\partial}{\partial y} + xz\frac{\partial^2}{\partial z\partial y}$.

Putting these values in above equation and taking $\hbar^2 = 1$ according to Hatree Atomic unit, we get

$$= -1\left(y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial y\partial x} + zx\frac{\partial^2}{\partial y\partial z}\right]$$

$$- (-1)\left[zy\frac{\partial^2}{\partial x\partial z} - xy\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial x\partial y} + xz\frac{\partial^2}{\partial z\partial y} + x\frac{\partial}{\partial y}\right]$$

After cancelling out equivalent terms,

$$= -\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right] = \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

In order to make it a physical significant entity we multiply $[\hbar = 1]$ & multiply and divide i

$\Rightarrow$ Above eq^n becomes $i\frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = iL_z$

$$\Rightarrow \boxed{[L_x, L_y] = i\hat{L}_z}$$

Similarly, $[L_y, L_z] = i\hat{L}_x$ & $[L_z, L_x] = i\hat{L}_y$

Hence proved

$[L_x, L_y], [L_y, L_z], [L_z, L_x] \neq 0$.

Hence they do not commute.

Q. Prove $[P_x, P_y] = 0$.

Ans: Proof $P_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $P_y = \frac{\hbar}{i}\frac{\partial}{\partial y}$

$$[P_x, P_y] = P_x P_y - P_y P_x$$

$$= \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

$$= \frac{\hbar^2}{i}\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right) - \frac{\hbar^2}{i}\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}\right)$$

$$= 0.$$

Q. Prove that $[L_y, L_x] \neq 0$

Proof: $L_y L_x - L_x L_y$

$$L_y L_x = \frac{1}{i^2}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -\left[z\frac{\partial}{\partial x}\cdot x\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\cdot y\frac{\partial}{\partial x} - z\frac{\partial}{\partial x}\cdot y\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\cdot x\frac{\partial}{\partial y}\right]$$

$$= -\left[z\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial x\,\partial y} - 0 - zy\frac{\partial^2}{\partial x^2} - 0 - x^2\frac{\partial^2}{\partial x\,\partial y} + 0 + xy\frac{\partial^2}{\partial z\,\partial x}\right]$$

$$= -\left[z\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial x\partial y} - zy\frac{\partial^2}{\partial x^2} - x^2\frac{\partial^2}{\partial x\partial y} + xy\frac{\partial^2}{\partial x\partial x}\right]$$

$$L_x L_y = \frac{1}{i^2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\right)$$

$$= -\left[x\frac{\partial}{\partial y}\cdot z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\times z\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\cdot x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right]$$

$$= -\left[xz\frac{\partial^2}{\partial y\partial x} - 0 - yz\frac{\partial^2}{\partial z^2} - 0 - x^2\frac{\partial^2}{\partial x\cdot\partial y} + 0 - y\frac{\partial}{\partial x} + yx\frac{\partial^2}{\partial x}\right]$$

$$= -\left[xz\frac{\partial^2}{\partial y\partial x} - y^2\frac{\partial^2}{\partial x^2} - x^2\frac{\partial^2}{\partial y\cdot\partial x} - y\frac{\partial}{\partial x} + yx\frac{\partial}{\partial x\cdot\partial x}\right]$$

$$\therefore [L_y, L_x] = L_y L_x - L_x L_y = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

$$= i\times\left[\frac{1}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$= i\hat{L}_x$$

→ This one here represents $\hbar$ in Hatree unit

Q. Prove that $[\hat{L}_y, \hat{L}_z] = i\hat{L}_y$

$[\hat{L}_y, \hat{L}_z] = [\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y]$

$\hat{L}_y\hat{L}_z = \frac{1}{i^2}\left(y\frac{\partial}{\partial z} - x\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$= -\left(y\frac{\partial}{\partial x}\cdot x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y}\cdot x\frac{\partial}{\partial y} + x\frac{\partial}{\partial y}\cdot y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\cdot y\frac{\partial}{\partial x}\right)$

$= -\left(xy\frac{\partial^2}{\partial x\cdot\partial y} - xx\frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} - y^2\frac{\partial^2 x}{\partial x\partial x}\right)$

$= -\left(xy\cdot\frac{\partial^2}{\partial x\cdot\partial y} - xx\frac{\partial^2}{\partial y^2} - y^2\frac{\partial^2}{\partial x\cdot\partial x} + x\frac{\partial}{\partial x}\right)$

$\hat{L}_z\hat{L}_y = \frac{1}{i^2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$

$= -\left(x\frac{\partial}{\partial y}\cdot y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\cdot y\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y}\cdot x\frac{\partial}{\partial y}\right)$

$= -\left(x\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial x\cdot\partial x} - xx\frac{\partial^2}{\partial y^2} + xy\frac{\partial^2}{\partial x\partial y}\right)$

$= -\left(x\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial x\partial x} - xx\frac{\partial^2}{\partial y^2} + xy\frac{\partial^2}{\partial y\cdot dx}\right)$

$\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = \left(x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\right)$

Multiplying & deviding with 'i' & using Hatree unit value $\hbar = 1$, we get

$[\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y] = i\left(\frac{1}{i}\right)\left(x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\right)$

$= iL_y \neq 0.$

$\therefore [\hat{L}_y, \hat{L}_z] \neq 0$, i.e they donot commute.

Q. Prove that :- $[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

Proof :-

$L^2 L_z - L_z L^2$

$= (L_x^2 + L_y^2 + L_z^2)L_z - L_z(L_x^2 + L_y^2 + L_z^2)$

(A) $\Rightarrow [\hat{L}_y^2, \hat{L}_z] = L_y^2 L_z - \hat{L}_z L_y^2$

$= \hat{L}_y\cdot\hat{L}_y\cdot\hat{L}_z - L_z L_y L_y$

By Adding $L_x L_z L_x$ & Substracting $L_x L_z L_x$ from the equation, we get

$$\hat{L}_x \hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y \hat{L}_x + \hat{L}_x \hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z \hat{L}_x$$

$$= \hat{L}_x \hat{L}_y \hat{L}_z - \hat{L}_x \hat{L}_z \hat{L}_x + \hat{L}_x \hat{L}_z \hat{L}_x - \hat{L}_z \hat{L}_y \hat{L}_x$$

$$= \hat{L}_x (\hat{L}_x \hat{L}_z - \hat{L}_z \hat{L}_x) + (\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y) \hat{L}_x$$

$$= \hat{L}_x (-i\hat{L}_y) + (iL_y)\hat{L}_x$$

$$= -i(\hat{L}_y \hat{L}_y + \hat{L}_y \hat{L}_x) \quad \text{--- (i)}$$

Similarly,

$$[\hat{L}_y^2 \hat{L}_x] = \hat{L}_y^2 \hat{L}_x - L_x L_y^2$$

$$= \hat{L}_y \hat{L}_y \hat{L}_x - L_x L_y L_y$$

Adding and Substracting $L_y L_z L_y$ to the above eq^n, we get

$$\hat{L}_y \hat{L}_y \hat{L}_x - \hat{L}_x \hat{L}_y \hat{L}_y + \hat{L}_y \hat{L}_z \hat{L}_y - \hat{L}_y \hat{L}_z \hat{L}_y$$

$$= \hat{L}_y \hat{L}_y \hat{L}_x - \hat{L}_y \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \hat{L}_y - \hat{L}_x \hat{L}_y \hat{L}_y$$

$$= \hat{L}_y [\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y] + [\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y] \hat{L}_y$$

$$= \hat{L}_y i\hat{L}_x + i\hat{L}_x \hat{L}_y$$

$$= i(\hat{L}_y \hat{L}_x + \hat{L}_x \hat{L}_y) \quad \text{--- (ii)}$$

By the property of commutators,

$$[\hat{L}_x^2, \hat{L}_x] = \hat{L}_x^2 \hat{L}_x - \hat{L}_x L_x^2$$

$$= L_x L_x L_x - L_x L_x L_x = 0 \quad \text{--- (iii)}.$$

On Adding (i), (ii) & (iii) we get

$$[-iL_y L_y - iL_x L_y + iL_y L_x + iL_x L_y] = 0$$

(B) $[L^2, L_x] = [\hat{L}_x^2 L_x] + [L_y^2, L_x] + [L_z^2 L_x]$

$$[L_x^2 L_x] = \hat{L}_x . \hat{L}_x . L_x - \hat{L}_x . \hat{L}_x \hat{L}_x = 0 \quad \text{--- (i)}$$

$$[L_y^2, L_x] = \hat{L}_y . \hat{L}_y \hat{L}_x - \hat{L}_x \hat{L}_y . \hat{L}_y$$

$$= \hat{L}_y \hat{L}_y \hat{L}_x - \hat{L}_y \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x \hat{L}_y - \hat{L}_x \hat{L}_y L_y$$

(Adding & Substracting $L_y L_x L_y$)

$= \hat{L}_y[\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y] + [\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y]\hat{L}_y$

$= \hat{L}_y[-i\hat{L}_z] + (i\hat{L}_z)L_y$

$= -i[\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y]$ —— (ii)

$[L_z^2, L_y] = \hat{L}_z\hat{L}_z L_x - \hat{L}_x\hat{L}_z L_z$

$= \hat{L}_z\hat{L}_z\hat{L}_x - \hat{L}_z\hat{L}_x\hat{L}_z + \hat{L}_z\hat{L}_x\hat{L}_z - \hat{L}_x\hat{L}_z\hat{L}_z$

$= L_z[\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z] + [\hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z]\hat{L}_z$

$= L_z[iL_y] + [iL_y]L_z$

$= i[\hat{L}_z L_y + \hat{L}_y L_z)$ —— (iii)

Adding (i) (ii) & (iii) we get:

$$[L^2, \hat{L}_x] = 0.$$

(c) $[L^2, L_y] = 0$

$[L^2, L_y] = [L_x\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\hat{L}_x] + [L_y^2, L_y] + [L_z^2, L_y]$

$[L_x^2\hat{L}_y] = \hat{L}_x\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\hat{L}_x$

$= \hat{L}_x\hat{L}_x\hat{L}_y - \hat{L}_x\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y\hat{L}_x - L_y\hat{L}_x\hat{L}_x$

$= \hat{L}_x[\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x] + [\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x]\hat{L}_x$

$= L_x[iL_z] + [iL_z]L_x$

$= i[L_xL_z + L_zL_x]$ —— (i)

$[\hat{L}_y^2\hat{L}_y] = \hat{L}_y\hat{L}_y\hat{L}_y - \hat{L}_y\hat{L}_y\hat{L}_y = 0$ —— (ii)

$[\hat{L}_z^2, \hat{L}_y] = \hat{L}_z\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z\hat{L}_z$

$= \hat{L}_z\hat{L}_z\hat{L}_y - \hat{L}_z\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y\hat{L}_z - \hat{L}_y\hat{L}_z\hat{L}_z$

$= \hat{L}_z[\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z] + [\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z]\hat{L}_z$

$= \hat{L}_z[-i\hat{L}_x] + [-i\hat{L}_x]\hat{L}_z$

$= -i[L_zL_x + \hat{L}_x\hat{L}_z]$

Adding (i), (ii) & (iii)

$$[L^2, \hat{L}_y] = 0$$

STEP UP AND STEP DOWN OPERATOR FOR ANGULAR - MOMENIUM

* $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ → step up operator.

$L_- = \hat{L}_x - iL_y$ → step down operator.

$\hat{L}_z \hat{L}_+ = \hat{L}_z(\hat{L}_x + i L_y)$ — (i)

$= \hat{L}_z \hat{L}_x + i\hat{L}_z \hat{L}_y$

we know that, $\hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z = iL_y$

$\Rightarrow \hat{L}_z \hat{L}_x = iL_y + \hat{L}_x L_z$ — (a)

Similarly, $\hat{L}_y \hat{L}_z - \hat{L}_z \hat{L}_y = i\hat{L}_x$

$\Rightarrow \hat{L}_y \hat{L}_z - iL_x = \hat{L}_z \hat{L}_y$

$\Rightarrow \hat{L}_z \hat{L}_y = \hat{L}_y \hat{L}_z - iL_x$ — (b)

Putting the values from eqn (a) & (b) in eqn (i), we get that

$\hat{L}_z \hat{L}_+ = iL_y + L_x L_z + i(\hat{L}_y \hat{L}_z - iL_x)$

$= iL_y + \hat{L}_x \hat{L}_z + i\hat{L}_y \hat{L}_z + \hat{L}_x$

$= L_x + i\hat{L}_y + (\hat{L}_x + iL_y)\hat{L}_z$

$= L_+ + L_+ L_z$

$\Rightarrow \boxed{\hat{L}_z \hat{L}_+ = L_+(1 + L_z)}$

$L_z L_- = \hat{L}_z(\hat{L}_x - i\hat{L}_y)$

$= \hat{L}_z \hat{L}_x - i\hat{L}_z \hat{L}_y$ — (ii)

Substituting values of $\hat{L}_z \hat{L}_x$ and $\hat{L}_z \hat{L}_y$ from the equation (a) & (b) in eqn (ii), we get

$i\hat{L}_y + \hat{L}_x L_z - i(\hat{L}_y \hat{L}_z - i\hat{L}_x)$

$= i\hat{L}_y + \hat{L}_x \hat{L}_z - i\hat{L}_y \hat{L}_z - \hat{L}_x$

$= i\hat{L}_y - \hat{L}_x + \hat{L}_x \hat{L}_z - i\hat{L}_y \hat{L}_z$

$= -1(\hat{L}_x - iL_y) + \hat{L}_z(\hat{L}_x - iL_y)$

$= -(L_-) + L_z L_- = L_-(L_z - 1)$

$\Rightarrow \boxed{\hat{L}_z L_- = L_-(L_z - 1)}$

There by the the relations we get are,

1. $\hat{L}_z \hat{L}_+ = L_+ (L_z + 1)$

2. $\hat{L}_z \hat{L}_- = L_- (L_z - 1)$

* $L_z L_+ (Y_l^m(\theta,\phi)) = L_+ (1 + L_z) Y_l^m(\theta,\phi)$
$= L_+ (1 + m) Y_l^m(\theta,\phi)$

$\Rightarrow L_z L_+ Y_l^m(\theta,\phi) = (m+1) L_+ Y_{lm}$

Similarly, $L_z L_- Y_{l,m} = L_- (L_z - 1) Y_{l,m}$
$= L_- (m-1) Y_{lm}$

$\Rightarrow L_z L_- Y_l^m(\theta,\phi) = (m-1) L_- Y_l^m(\theta,\phi)$

* The $\hat{L}_+ Y_{l,m}$ is the eigen vector of $\hat{L}_z$ with eigen value $(m+1)$, one unit vector and $\hat{L}_- Y_{l,m}$ is an eigen vector of $\hat{L}_z$ with eigen value $(m-1)$ and unit vectors.

$$L_+ \hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y)$$
$$= \hat{L}_x^2 + i\hat{L}_y\hat{L}_x - i\hat{L}_x\hat{L}_y + \hat{L}_y^2$$
$$= \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y)$$
$$= \hat{L}_x^2 + \hat{L}_y^2 + (-iL_z)i$$
$$= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z$$

$$\hat{L}_+ \hat{L}_- = \hat{L}_x^2 + L_y^2 + \hat{L}_z$$
$$= L_x^2 + L_y^2 + L_z^2 + \hat{L}_z - \hat{L}_z^2$$
$$= L^2 - L_z^2 + \hat{L}_z$$

$$\Rightarrow \boxed{L^2 = \hat{L}_+ L_- + L_z^2 - \hat{L}_z}$$

$* \ \hat{L}^2 = \hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hat{L}_z$

Proof :- $\hat{L}_- \hat{L}_+ = (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y)$

$= \hat{L}_x^2 + \hat{L}_y^2 - i\hat{L}_y\hat{L}_x + i\hat{L}_x\hat{L}_y$

$= \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)$

$= \hat{L}_x^2 + \hat{L}_y^2 + i(iL_z)$

$= L_x^2 + L_y^2 - \hat{L}_z$

$= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 - L_z - L_z^2$

$= \hat{L}^2 - \hat{L}_z^2 - \hat{L}_z$

$\Rightarrow \boxed{L^2 = \hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hat{L}_z}$

Q. Show that $[L^2, L_+] = 0$.

$L^2 = L_x^2 + L_y^2 + L_z^2 \ , \ L_+ = L_x + iL_y$

$[L^2, L_+] = L^2 L_+ - L_+ L^2 = [(L_x^2 + L_y^2 + L_z^2)(L_x - iL_y) - (L_x + iL_y)(L_x^2 + L_y^2 + L_z^2)]$

As, $[L_x^2, L_z] = 0$,

Therefore, $[L^2, L_+]$ or $[L^2, L_-] = 0$.

Q. Show that $[L_z, L_+] = 1 . L_+$

$[L_z, L_x] + i[L_z, L_y]$

$= iL_y + i(-iL_x)$

$= iL_y + L_x$

$= L_x + iL_y$

$= 1 . L_+$

Q. Show that $[L_z, L_-] = -1 . L_-$

$[L_z, L_-] = [L_z, L_x] - i[L_z, L_y]$

$= iL_y - i[-iL_x] = iL_y - L_x$

$= -1(L_x - iL_y) = -1 . L_-$.

* $L_+ Y_{l,m} = C_+ Y_{l,m}$ and $L_- Y_{l,m} = C_- Y_{l,m}$

To find the values of C_+ & C_-, we have to first normalize the equations

$$L_+ Y_{l,m} = C_+ Y_{l,m}$$

$$\Rightarrow \langle L_+ Y_{l,m} | L_+ Y_{l,m} \rangle = 1$$

$$\Rightarrow L_- L_+ \langle Y_{l,m} | Y_{l,m} \rangle = |C_+|^2 \langle Y_{l,m} | Y_{l,m} \rangle$$

$$\Rightarrow (L^2 - L_z - L_z^2) \langle Y_{l,m} | Y_{l,m} \rangle = |C_+|^2 \langle Y_{l,m} | Y_{l,m} \rangle$$

Replacing $L_- L_+ = L^2 - L_z^2 - L_z$

substituting the corresponding eigen values

$$\Rightarrow (l(l+1) - m - m^2) \langle Y_{l,m} | Y_{l,m} \rangle = C_+^2 \langle Y_{l,m} | Y_{l,m} \rangle$$

$$\Rightarrow C_+ = \sqrt{l(l+1) - m - m^2}$$, { $\langle Y_{l,m} | Y_{l,m} \rangle$ is normalized i.e 1

$$\Rightarrow \boxed{C_+ = \sqrt{l(l+1) - m(m+1)}}$$

Similarly, $\langle L_- Y_{l,m} | L_- Y_{l,m} \rangle = C_-^2 \langle Y_{l,m} | Y_{l,m} \rangle$

$$\Rightarrow \langle L_+ Y_{l,m} | L_- Y_{l,m} \rangle = C_-^2 \langle Y_{l,m} | Y_{l,m} \rangle$$

$$\Rightarrow L_+ L_- \langle Y_{l,m} | Y_{l,m} \rangle = C_-^2 \langle Y_{l,m} | Y_{l,m} \rangle$$

substituting the value of $L_+ L_-$

$$(L^2 - L_z^2 + L_z) \langle Y_{l,m} | Y_{l,m} \rangle = C_-^2$$

$$\Rightarrow (l(l+1) - m^2 + m) \langle Y_{l,m} | Y_{l,m} \rangle = C_-^2$$

$$\Rightarrow C_- = \sqrt{l(l+1) - m(m-1)} .$$

Q11 Show that $l \geq m$

We know $L^2 = L_x^2 + L_y^2 + L_z^2$

$$\Rightarrow L^2 = L_x^2 + L_y^2 + L_z^2$$

$$L_x^2 + L_y^2 = L^2 - L_z^2$$

$$(L^2 - L_z^2)\, Y_{l,m} = [l(l+1) - m^2]\, \hbar\, Y_{l,m}$$

$\Rightarrow l(l+1) - m^2 > 0$ as sum of two squared terms is always positive.

$\Rightarrow l \geq |m|$