BBA-11

Block - 4

ଓଡ଼ିଶା ରାଜ୍ୟ ମୁକ୍ତ ବିଶ୍ୱବିଦ୍ୟାଳୟ, ସମ୍ବଲପୁର

Odisha State Open University

Sambalpur

BBA

BACHELOR OF BUSINESS ADMINISTRATION

Quantitative Techniques for Management

DECISION THEORY, GAME THEORY & SIMULATION

ଓଡ଼ିଶା ରାଜ୍ୟ ମୁକ୍ତ ବିଶ୍ୱବିଦ୍ୟାଳୟ, ସମ୍ବଲପୁର, ଓଡ଼ିଶା
Odisha State Open University, Sambalpur, Odisha
Established by an Act of Government of Odisha.

Bachelor of Business Administration (BBA)

Quantitative Techniques for Management
BBA-11

Block

4 DECISION THEORY, GAME THEORY & SIMULATION

Unit-1
Introduction to Decision Theory

Unit-2
Elements of Decision Theory and Markov Chains

Unit-3
Game Theory

Unit-4
Simulation

BBA – Course - 11
Block - 4

1st Edition – 2021

Course Writers : **Dr. Prafulla Kumar Swain**
Assistant Professor
Department of Statistics
Utkal University, Vani Vihar
(Unit-1 and Unit-2)

Dr. Jyotiranjan Nayak
Associate Professor
ICFAI Foundation for Higher Education, Hyderabad
(Unit-3 and Unit-4)

Course Editors : **Dr. Ansuman Jena**
Academic Consultant (Management)
Odisha State Open University, Sambalpur
(Unit-3 and Unit-4)

Dr. Kishore Kumar Das
Associate Professor and Head
Department of Commerce
Ravenshaw University, Cuttack
(Unit-1 and Unit-1)

Programme Coordinator: **Dr. Ansuman Jena**
Academic Consultant (Management)
Odisha State Open University
Sambalpur

Formatting: **Lipina Sahoo**
Jr. Executive (Non-Academic
Odisha State Open University
Sambalpur

Material Production

Dr. Manas Ranjan Pujari
Registrar
Odisha State Open University
Sambalpur

Unit-1
Introduction to Decision Theory

Structure:

1.1. Learning Objectives

The main aim of this unit is to give basic idea on the concept of decision theory and its importance in decision management. On completion of this unit the reader can able to answer the following questions:

- What is decision theory?
- How it works in decision management?
- What is a payoff table?
- What is opportunity loss table?

1.2. Introduction

Every day we come across with different situations where we need to take a decision from two or many alternatives. Before taking the decision, we need be more careful as it affects our day-to-day life immediately or/and later. The decisions may be on, whether to take admission in management course in regular mode or distance mode, where to stay at hostel or rent, from where the company purchase its raw material etc.

The goal of decision theory is to assist people and organizations in making decisions. It gives a useful conceptual framework for making critical decisions. Choosing of the best act from multiple choices under specific circumstances is referred to as "decision making" or "decision taking".

1.3. What is decision theory?

The decision theory concerned with several practical decision problems where the decision maker has to take a decision from several alternative options under wide verity of situation over which he/she has no control. Different statistical techniques viz.

probability theory under uncertainty and risk helps in selecting the optimal course of action.

In the same light the decision theory also well known as "Statistical decision theory". The statistical decision theory attempts to provide the logical structure into different actions, states of nature, possibilities and expected payoffs from each of these possible outcomes.

1.4. Terminology Involved in Decision Theory

Let us discuss some basic terminology which will ultimately help to understand the concept in more detail.

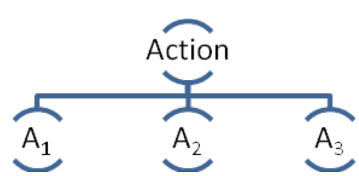
1.4.1. Decision / Course of action

Generally, a single decision is made from a set of predefined alternative course of actions. This is also known as strategies, act or actions etc.

Let the action space be A with three different possible actions a_1, a_2, $a_3$denoted as; A= $\{a_1, a_2, a_3\}$ or $\{A_1, A_2, A_3\}$. This action space can also be represented by a matrix form (either in a row or in a column) and by a tree diagram as follows;

Actions		
A_1		
A_2		
A_3		

Actions	A_1	A_2	A_3



1.4.2. Conflict

As per the differential in behavior and mentality, a group of people in an organization or between two departments there exist a conflict between them which might restrict them to reach their goal. These conflicts should be minimized by the decision maker in bringing full fledge success of the individual or organization.

1.4.3. Decision maker

The decision makers are the individual or the group of individuals who made an appropriate choice among all the available options.

1.4.4. Event / State of nature

The event identifies the occurrences which are beyond the control of decision makers and that determine the amount of success for a particular set. These are also known as state of nature or outcome. The set of events can be represented as follows:

$$E = \{E_1, E_2, E_3, \ldots\ldots\ldots, E_n\}$$

$$\text{or } S = \{S_1, S_2, S_3, \ldots\ldots\ldots, S_n\}$$

$$\text{or } \Omega = \{\theta_1, \theta_2, \theta_3, \ldots\ldots\ldots, \theta_n\}$$

1.4.5. Payoffs

The outcome of the combination of an action with each possible event is said to be an outcome. The short-lived loss or gain of every such outcome is said to be payoff. The payoff can be considered as the quantitative form of saving time and cost.

1.4.6. Regret (Opportunity Loss)

Regret or opportunity loss is the difference between the maximum conceivable profit for an event and the actual profit gained for the particular action done. This loss incurred due to not taking best possible course of action.

Note:

The chosen actions and their correspondent events can be shown by the earlier discussed methods i.e., matrix (row and column) method and tree method etc.

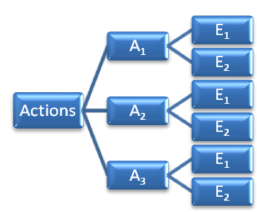
Row method

Events→ Actions↓	E_1	E_2	…
A_1			
A_2			
A_3			

Column method

Actions → Events ↓	A_1	A_2	A_3
E_1			
E_2			
⋮			

Tree method



1.5. Payoff table

Let us consider there are n alternative actions and m possible events (state of nature), thus there are n×m possible outcomes or payoffs. These n×m payoffs can be represented in a tabular form known as payoff table or payoff matrix.

Payoff Table					
Events ↓	Alternative Actions →				
	A_1	A_2	A_3	…	A_n
E_1	a_{11}	a_{12}	a_{13}	…	a_{1n}
E_2	a_{21}	a_{22}	a_{23}	…	a_{2n}
E_3	a_{31}	a_{32}	a_{33}	…	a_{3n}
⋮	⋮	⋮	⋮	⋮	⋮
E_m	a_{m1}	a_{m2}	a_{m3}	…	a_{mn}

Where; a_{ij}= The payoff of the i^{th} event and j^{th} action.

Example-1

Consider a shop keeper who purchase a weekly science magazine for 8 rupees and sells the same for 10 rupees. At the end of the week, the unsold copies of the science magazine are disposed for 3 rupees each. According to past experience the weekly demand for the science magazine is between 70 and 75 copies. Assuming that the order for this particular science magazine can be placed only once during the week the problem before the shop keeper is to decide how many copies of science magazine should be purchased for the next week. Now construct a payoff table for the above case.

Answer:

In this problem the weekly demand varies between 70 to 75. So, the possible events and possible actions are as follows:

Events↓	Actions↓
E_1= Demand of 70 copies E_2= Demand of 71 copies E_3=Demand of 72 copies E_4= Demand of 73 copies E_5= Demand of 74 copies and E_6= Demand of 75 copies	A_1= Purchase of 70 copies A_2= Purchase of 71 copies A_3= Purchase of 72 copies A_4= Purchase of 73 copies A_5= Purchase of 74 copies and A_6= Purchase of 75 copies

Let Q be the quantity per unit for the science magazine, D be the demand in unit of the science magazine. Now the profit can be calculated as:

$$\text{Profit} = \begin{bmatrix}\text{Total amount from the sale} \\ \text{of all copies that were purchased}\end{bmatrix} - \begin{bmatrix}\text{total cost of purchasing the copies} \\ \text{of all the science magazine}\end{bmatrix}$$

The profit depends upon Q and D as follows;

$$P = \begin{cases} (10Q - 8Q) = 2Q ; & \text{If } D \geq Q \\ [10D + 3(Q - D)] - 8Q = 7D - 5Q; & \text{If } D < Q \end{cases}$$

Hence, there are 36 different payoffs for 36 different combinations of events and actions as in the following table;

$(E_1, A_1) = 2 \times 70 = 140$	$(E_2, A_1) = 2 \times 70 = 140$	⋮	⋮	⋮	$(E_6, A_1) = 2 \times 70 = 140$
$(E_1, A_2) = 7\times70 - 5\times71 = 130$	$(E_2, A_2) = 2\times71 = 142$	⋮	⋮	⋮	$(E_6, A_2) = 2\times71 = 142$
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
$(E_1, A_6) = 7 \times 70 - 5\times75 = 115$	$(E_2, A_6) = 7\times71 - 5\times75 = 122$	⋮	⋮	⋮	$(E_6, A_6) = 2 \times 75 = 150$

Hence, payoff table can be constructed using the above payoffs for all the combinations.

Payoff Table						
Events ↓	Alternative Actions →					
	$A_1 = 70$	$A_2 = 71$	$A_3 = 72$	$A_4 = 73$	$A_5 = 74$	$A_6 = 75$
$E_1 = 70$	140	135	130	125	120	115
$E_2 = 71$	140	142	137	132	127	122
$E_3 = 72$	140	142	144	139	134	129
$E_4 = 73$	140	142	144	146	141	136
$E_5 = 74$	140	142	144	146	148	143
$E_6 = 75$	140	142	144	146	148	150

Example-2

A farmer can grow any one of three crops in his field. The yield of each crop depends on3 different weather condition. Construct a payoff table showing payoff in each case if prices of the three products are as indicated in the last row of the yield matrix.

	Weather condition			
Yield in kg Per hector		Paddy (A_1)	Ground Nut (A_2)	Tobacco (A_3)
	Dry (E_1)	500	800	100
	Moderate (E_2)	1700	1200	300
	Damp (E_3)	4500	1000	200
	Price per kg	1.25	4.00	15.00

Answer:

Payoff Table			
Events ↓	**Alternative Actions →**		
	A_1	**A_2**	**A_3**
E_1	500×1.25=625	800×4=3200	100×15=1500
E_2	1700×1.25=2125	1200×4=4800	300×15=4500
E_3	4500×1.25=5625	1000×4=4000	200×15=3000

1.6. Opportunity Loss (Regret) Table

Let us consider a payoff table with its payoff values a_{11}, a_{12}, a_{13}, ….,a_{mn} (a_{ij} be the payoff of the i^{th} event and j^{th} action) and M_{ij}be the maximum payoff corresponding to n actions and m events. If A_1 action is taken then the loss of opportunity becomes M_{1j}-a_{1j} and so on. Now the opportunity loss can be calculated as given in the following table:

Opportunity loss Table					
Events ↓	**Alternative Actions →**				
	A_1	**A_2**	**A_3**	**…**	**A_n**
E_1	$M_{11}-a_{11}$	$M_{12}-a_{12}$	$M_{13}-a_{13}$	**…**	$M_{1n}-a_{1n}$
E_2	$M_{21}-a_{21}$	$M_{22}-a_{22}$	$M_{23}-a_{23}$	**…**	$M_{21}-a_{2n}$
E_3	$M_{31}-a_{31}$	$M_{32}-a_{32}$	$M_{33}-a_{33}$	**…**	$M_{3n}-a_{3n}$
⋮	⋮	⋮	⋮	⋮	⋮
E_m	$M_{m1}-a_{m1}$	$M_{m2}-a_{m2}$	$M_{m3}-a_{m3}$	**…**	$M_{mn}-a_{mn}$

Example: 3

Let us consider the payoff table of the example 1. We observe that the demand is of 70 copies of the science magazine, then the optimal act is to buy 70 copies to get a profit of 140 rupees. In the first row the maximum profit is 140 rupees corresponding to action A_1. If any alternative strategies are adopted then the profit earned would be less and the greater the departure from the optimal strategy the lesser the profit earned and consequently the greater the opportunity loss. With the strategy of buying 74 copies the profit is 120 which is 20 less than the profit that could be earned at the level of demand.

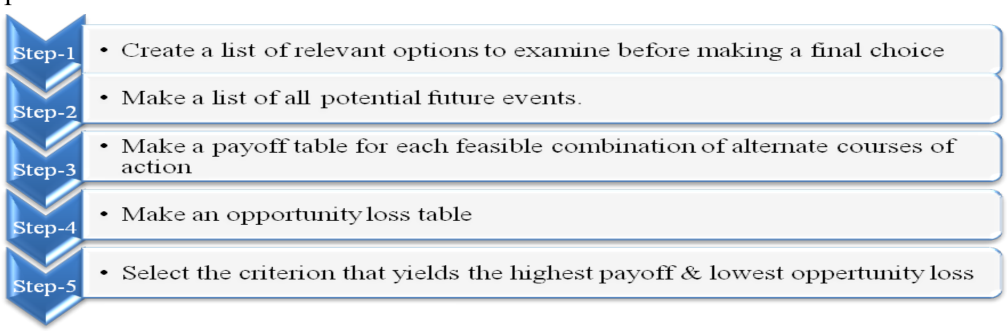
Answer

If the demand turnout for the 72 copies the optimal course would be the order of 72 copies, the profit would be 144.Any order size other than this causes lesser profit than this. The opportunity loss table is obtained by subtracting all the values of a row from the highest profit value of that row as follows.

Opportunity loss Table						
Events ↓	**Alternative Actions →**					
	$A_1 = 70$	**$A_2 = 71$**	**$A_3 = 72$**	**$A_4 = 73$**	**$A_5 = 74$**	**$A_6 = 75$**
$E_1 = 70$	140–140	140–135	140–130	140–125	140–120	140–115
$E_2 = 71$	142–140	142–142	142–137	142–132	142–127	142–122
$E_3 = 72$	144–140	144–142	144–144	144–139	144–134	144–129
$E_4 = 73$	146–140	146–142	146–144	146–146	146–141	146–136
$E_5 = 74$	148–140	148–142	148–144	148–146	148–148	148–143
$E_6 = 75$	150–140	150–142	150–144	150–146	150–148	150–150

1.7. Different Steps in Decision Making

The following steps can be followed in making a good decision using decision theory approach:



Example-4

A restaurant prepares a special type of curry at total average cost of 4 rupees per plate and sells it at a price of 6 rupees per plate. The curry is prepared in the morning and is sold during the same day. Unsold curry during the same day is spoiled and is to be thrown away. According to the past sale, number of plates repaired is not less than 50 or greater than 53. Formulate the (a) action space, (b) state of nature space, (c) payoff table and (d) loss table.

Answer:

a) The restaurant will not prepare less than 50 plate or more than 53 plates. Thus, the act or act or courses of action open to him are:

A_1: Prepare 50 plates,

A_2: Prepare 51 plates,

A_3: Prepare 52 plates,

A_4: Prepare 53 plates

Thus, the action space is $A= \{A_1, A_2, A_3, A_4\}$

b) The state of nature is daily demanded for curry. Then are four possible states of nature being:

E_1: Demand 50 plates,

E_2: Demand 51 plates,

E_3: Demand 52 plates,

E_4: Demand 53 plates

Thus, the state of nature space is $E= \{E_1, E_2, E_3, E_4\}$

c) The uncertainty element in the given problem is the daily demand. The profit of the restaurant is subject to the daily demand.

Let n=quantity demand and m=quantity produced.

For $n \geq m$; profit = (Selling price – Cost price) × m

$= (6-4) \times m$

$=2m$

For $n \leq m$; profit = {(Selling price – Cost price) × n} – {Cost price × (m–n)}

$=2n - 4(m-n)$

$=6n-4m$

Payoff Table				
Demand (n) ↓	**Supply (m) →**			
	A_1 =50	**A_2 =51**	**A_3 =52**	**A_4=53**
E_1 =50	100	96	92	88
E_2 =51	100	102	98	94
E_3 =52	100	102	104	100
E_4=53	100	102	104	106

d) For calculating the opportunity loss, we need to determine the maximum payoff in each state of nature. In this state;

1st maximum payoff = 100

2nd maximum payoff = 102

3rd maximum payoff = 104

4th maximum payoff = 106

Now the opportunity loss table corresponding to above payoff table is given by;

	Opportunity loss Table			
Events ↓	**Alternative Actions →**			
	A_1 = 50	**A_2 = 51**	**A_3 = 52**	**A_4 = 53**
E_1 = 50	100–100=0	100–96=4	100–92=8	100–88=12
E_2 = 51	102–100=2	102–102=0	102–98=4	102–94=8
E_3 = 52	104–100=4	104–102=2	104–104=0	104–100=4
E_4 = 53	106–100=6	106–102=4	106–104=2	106–106=0

1.8. Summary

In this unit a detailed process of making a decision from available alternatives has been discussed. This decision taking is also known as course of action, which is based on different criterion viz. payoff table, opportunity loss table etc.

1.9. Key Terms

Decision (Course of action), Decision theory, Conflict, Decision maker, Event (State of nature), Payoffs, Opportunity Loss (Regret), Payoff table and Opportunity Loss (Regret) Table.

1.10. Check Your Progress

Choose the correct answer

1) Decision analysis is concerned with;
   (a) Determining optimal decision in sequential manner
   (b) Analysis of information that is available
   (c) Decision making under certainty
   (d) All of the above
2) Which of the following statement is not true?
   (a) Major decision are the decisions which involve large sum of money and require prior sanction
   (b) Strategic decisions are the decisions which have far reaching effect on the future course of action
   (c) Tactical decisions are the decisions taken by the executive in their individual capacity
   (d) Organizational decisions are the decision taken by the executives in their official capacity.
3) Which is not a type of decision-making environment?
   (a) Uncertainty
   (b) Certainty
   (c) Risk
   (d) None of the above
4) In the study of decision making under uncertainty;
   (a) Opportunity loss in the payoff matrix is also called conditional payoff
   (b) Events are also called states of nature
   (c) The decision maker has no idea about the possible states of nature and their probabilities.
   (d) all of the above

Answer

1. (d) 2. (c) 3. (d) 4. (d)

1.11.References

❖ Kalavathy, S (2013): Operations Research, 4th Edition, Vikas Publishing House, Pvt. Ltd.

❖ Murthy P. R. (2007): Operations Research, second edition, New Age International (P) Ltd., Publishers, ISBN (13): 978-81-224-2944-2.

❖ Swarup, K., Gupta, P. K., & Mohan, M. (2017): Operations Research, Sultan Chand & Sons, New Delhi.

- Gupta, R. K. (2010): Operations Research, Krishna Educational Publishers.
- Ravindran, Phillips and Solberg (2012): Operation Research: Principles and practice, Second edition, Wiley student edition.
- Taha, H. A. (2009): Operation research: An introduction, Eighth edition, Pearson Education Publication.
- Vohra, N. D.: Quantitative Management, Tata McGraw Hill
- Kapoor, V. K.: Operation Research, Sultan Chand & Sons
- Sharma, J. K.: Operation Research Theory &Applications, Macmillan India Limited

1.12.Model Questions

- A toy repairman has an opportunity to purchase a stock of discontinued toys. They were originally supposed to be sold for 400 rupees each. The repairman is offered all 5 toys for 500 rupees, which makes his cost for each toy 100 rupees. If he sells them, he believes he can get 250 rupees for each toy, by making a profit of 150 rupees. He has two options; either to buy all the discontinued toys or not to buy at all. There are 6 states of nature there and these being the demand for 0, 1, 2, 3, 4 and 5 toys. Hence,

  a) Prepare a payoff and a regrate table for this problem and

  b) If the repairman has the option of buying any number of toys from 0 to 5, find the average expected payoff and average expected regret for each action.

- A scientist makes a product whose principal ingredient is a chemical A. At the moment the scientist spends 1000 rupees per year on supply of A. But there is a possibility that the price may soon increase to four times of its present figure because of an unknown cause. There is another chemical B, which could be used in conjunction with third chemical C in order to give the same effect as chemical A. The chemical B and C would together cost around 3000 rupees per year. But their prices are unlikely to rise. What action should be taken by the scientist?

Unit-2
Elements of Decision Theory and Markov Chains

Structure :

2.1. Learning Objectives

The main idea behind this unit is to give a brief idea about decision environment. After completion of this unit the reader will be able to understand the following concepts;

- Decision environment
- Expected monetary value (EMV)
- EMV of perfect information
- EMV of sample information

2.2. Introduction to Elements of Decision Theory

Decision theory is every so often called Statistical decision theory or Bayesian decision theory. Its principal strengths are:

i) it gives a suitable model to the decision maker for an optimum decision in a situation where multiple states of nature are available,

ii) it also factors in the economic loss due to wrong decisions in that model and iii)it allows the decision maker to use prior information, which ultimately helps to make the best decision.

2.3. Environment of Decision Making

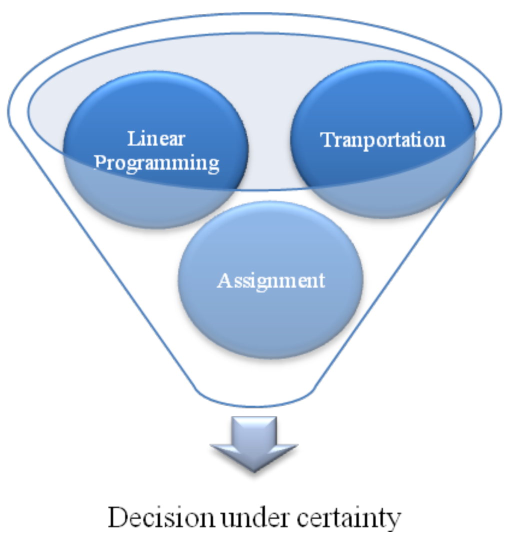
When a decision maker is met with several alternatives, decision analysis is used to decide the best course of action. In the same note we may come across with several situations known as decision environment that can be further divided into four types as follows:

a) Decision taking under certainty
b) Decision taking under conflict
c) Decision taking under uncertainty
d) Decision taking under risk

In this unit decision taking under risk will be explained in more details, however others are explained with its definition only.

2.3.1. Decision Taking Under Certainty

In this situation, the decision maker has total clarity about the implications of each decision option with certainty. Here certainty means there exists only one possible state of nature or outcome of the decision. The problems viz. linear programming, transportation, assignments and sequencing are the case of decision taking under certainty.



2.3.2. Decision Taking Under Conflict

In several situations, at the time of decision making the state of nature is neither completely certain nor completely uncertain. However, partial information is available and decision making under these circumstances is known as decision making under partial uncertainty or conflict.

For example, A situation where two or more companies are competing with each other and marketing the same or identical product. Decision making under conflict has its applications in competitive games as well.

2.3.3. Decision Taking Under Uncertainty

In decision making under uncertainty, only the payoffs are known and not the probability associated with any state of nature. This case arises when a totally new set of products is introduced in the market or a new company is setup to produce the product.

There are several decision approaches that can be used to solve a problem under uncertainty and they are: i) Optimistic approach (Maximax), ii) Pessimistic approach (Maximin), iii) Hurwicz approach (Realism) iv) Savage or Regret approach (Minimax), v) Baye's or Laplace approach (Equally likely) etc.

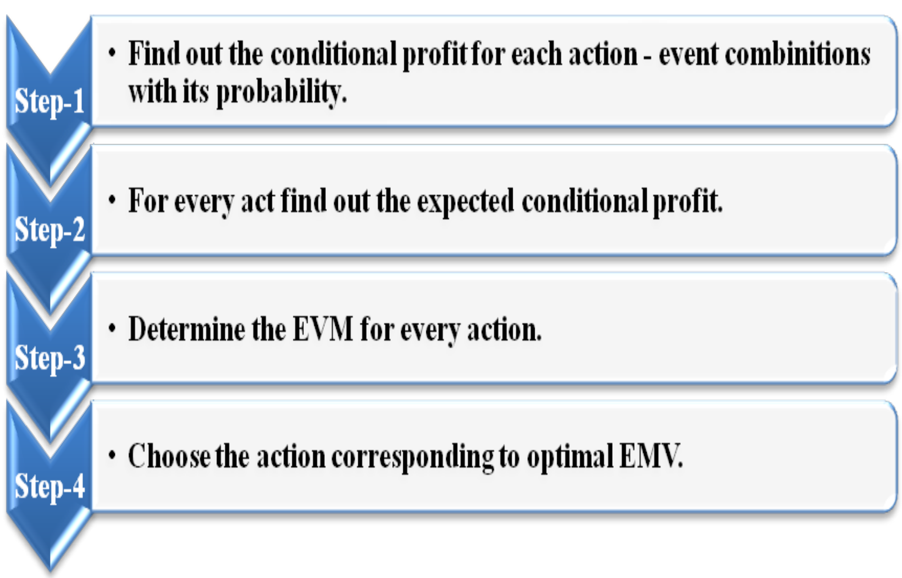
2.3.4. Decision Taking Under Risk

In this case the decision maker has to face different states of nature to make the decision. On the basis of prior knowledge or available records or past experiences he/she has to assign probability to the occurrence of each state of nature. On the basis of this probability distribution, the decision maker may select the best course of action having highest expected payoff value. Decision taking in this approach can be treated as the problem under risk.

Under this risk approach the decision can be taken by using the following decision criterion; i) Expected monetary value (EMV), ii) Expected opportunity loss (EOL) and iii) Expected value of perfect information (EVPI).

i) Expected monetary value (EMV)

The expected monetary value is usually used to evaluate the various courses of action. EMV for a course of action is nothing but the expected value of the conditional payoff. The calculation of EMV involves the following steps;



Example: 1

A woman has two choices either running of a coffee stall or a cold drink stall at a seabeach during the summer season. If it is a fairly cool summer, he should make 5000 rupees by running hand. If he operates the cold drink stall his profit is estimated at 6500 rupees if the summer is hot but only 1000 if it is cool. There is a 40% chance of the summer being hot.

Should he go for running the coffee stall or the cold drink stall? Give a suitable mathematical argument.

Solution:

Here, the following payoff table shows a case of conditional profit resulting from the described act and event combinations:

Event E_i (Weather)	Probability $P(E_i)$	Conditional payoff (Rs.) (Action)	
		Coffee Stall	Cold drink Stall
Cool Summer	0.6	5000	1000
Hot Summer	0.4	1000	6500

The expected conditional payoffs are computes as follows:

Event E_i (Weather)	Probability $P(E_i)$	Conditional payoff (Rs.) (Action)	
		Coffee Stall	Cold drink Stall
Cool Summer	0.6	5000	1000
Hot Summer	0.4	1000	6500
	EMV	3400	3200

As, the expected monetary value of running a coffee stall is higher than the cold drink stall, the woman should choose for running a coffee stall.

ii. Expected opportunity Loss (EOL)

Minimizing expected opportunity loss (EOL) is an alternative to maximizing expected money value (EMV).

The expected opportunity loss or expected value of regrets are assessed in the same way as expected payoffs using EMV criteria. Important steps of EOQ approach are stated as follows:

Step-1

Make a conditional profit table for each act-event combination with event probabilities

Step-2

Measure the conditional opportunity loss (COL) for each event by identifying the most favorable action having maximum payoff. Then take the difference between the conditional profit and every conditional profit for those events.

Step-3

For every action, calculate the expected conditional opportunity loss (COL). Then add these values to have expected opportunity loss (EOL) for those actions.

Step 4

The action with minimum COL value can be chosen.

Example: 2

An investor proposes the following investment alternatives with rates of return (in %).

Investment alternatives	State of return nature (Market conditions)		
	Low	Medium	High
Regular Shares	7 %	10 %	15 %
Risky Shares	-10 %	12 %	25 %
Property	-12 %	18 %	30 %

For the last 300 days, 150 days have been medium market conditions and 60 days have had high market conditions. Based on the above given data find an optimum investment strategy for the investment.

Solution:

From the given information, the probabilities of low, medium and high market conditions would be 90/300=0.3, 150/300=0.5 and 60/300=0.2 respectively. The expected payoffs for different alternatives are as follows;

Calculation of expected return				
Market conditions	Probability	Strategy		
		Regular shares	Risky shares	property
Low	0.3	0.07 × 0.3	-0.10 × 0.3	-0.12 × 0.3
Medium	0.5	0.10 × 0.5	0.12 × 0.5	0.18 × 0.5
High	0.2	0.15 × 0.2	0.25 × 0.2	0.30 × 0.2
	Expected Return	0.10	0.14	0.186

As the expected return is highest i.e., 18.6% for the property, the investor should invest in property.

iii. Expected value of perfect information (EVPI)

If we have perfect information before making a decision, the expected profit with perfect information is the predicted return in the long term. The largest amount one would be willing to pay to obtain perfect information about which event will occur may be described as the expected value of perfect information (EVPI). EPPI provides the highest EMV obtained with perfect knowledge of which event will occur. Let us consider EMV' as the highest obtainable EMV without perfect knowledge, then the perfect knowledge/ information will increase with expected profit from EMV' up to the value of EPPI, therefore the amount of increase becomes EVPI.

Now we get;

$$EVPI = EPPI - EMV'$$

Example: 3

A wholesaler of sports items has an opportunity to buy 5000 pairs of gloves that have been declared surplus by the sports authority. The wholesaler will pay 50 rupees per pair and can obtain 100 rupees per pair by selling gloves to retailers. The price is well established, but the wholesaler is in doubt as to just how many pairs he will be able to sell. Any gloves left over; he can sell to discount outlets at 20 rupees a pair. After a careful consideration of the historical data, the wholesaler assigns probabilities to the demand as follows:

Retailers Demand	Probability
1000 pairs	0.6
3000 pairs	0.3
5000 pairs	0.1

i) Compute the conditional monetary and expected monetary values.

ii) Compute the expected profit with a perfect predicting device.

iii) Compute the EVPI.

Solution:

From the given problem;

Cost price per pair	= 50 rupees
Selling price per pair	= 100 rupees
Profit per pair	= 50 rupees (if sold)
Disposal selling price	= 20 rupees (if unsold)
Loss on each unsold pair	= 50 - 20 = 30 rupees

Conditional profit values can be computed by using following method;

$$CP = \begin{cases} 50\,S; & \text{When } D > S \\ 50\,D - 30\,(S-D); & \text{When } D < S \end{cases}$$

Where; CP = Conditional Profit, D = Pairs Demanded and S = Pairs Stocked

i) The resulting conditional payoffs and corresponding expected payoffs are computed in the following table; (in 000′ rupees)

Retailer's demand	Probability	Conditional payoffs (Rs.) (Stock per week)			Expected payoffs (Rs.) (Stock per week)		
		1000 pairs	3000 pairs	5000 pairs	1000 pairs	3000 pairs	5000 pairs
1000 pairs	0.6	50	-10	-70	30	-6	-42
3000 pairs	0.3	50	150	90	15	45	27
5000 pairs	0.1	50	150	250	5	15	25
				EVM	50	54	10

ii) The expected profit under perfect information (EPPI) is computed below:

(in 000′ rupees)

Retailer's demand	Probability	Conditional payoffs (Rs.)			Under perfect information	
		1000 pairs	3000 pairs	5000 pairs	Maximum payoff	Expected payoffs
1000 pairs	0.6	50	-10	-70	50	30
3000 pairs	0.3	50	150	90	150	45
5000 pairs	0.1	50	150	250	250	25
					EPPI	100

iii) Now; EVPI = EPPI – EMV′ = 100 – 54 = 46 (in 000′)

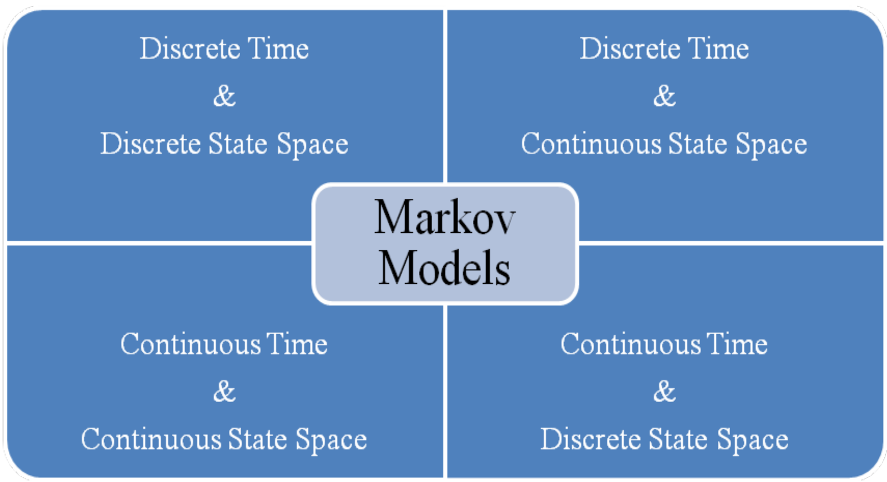
⇨ EVPI = 46,000 rupees /-

2.4. Introduction to Markov analysis and Notations

Markov analysis deals with analyzing the current movement of a particular variable with an effort to forecast its future movement. It has been used since past several years in many fields like marketing, accounting, maintenance policies, queueing system and job assignments.

The Markov process, a random evaluation of memoryless system, was developed by a Russian mathematician Andrey Markov. This was further developed by A. N. Kolmogorov and W. Feller. The Markov process has its applications in solving inventory problems, equipment replacement problems, plant location problems, brand switching problems and dynamic system problems.

Markov analysis is based on the following types of probabilistic models;



Let us consider a random process $\{X(t),\ t \epsilon T\}$, where T is the parameter set. The possible values assumed by $X(t)$ is known as state and the collection of all possible values of the state form a state space denoted by E.

If the parameter set T of a random process is discrete, then the process is called a discrete parameter or discrete time process denoted by$\{X_n, n = 1,2,...\}$. If T and E are both discrete then the random process is called a discrete random sequence.

For an illustration, if X_n represents the number of heads obtained in the nth toss of two fair coins, then $\{X_n, n \geq 1\}$is a discreate random sequence. Here, T= {1, 2…} and E= {0,1,2}

If T is discrete and E is continuous then the random process is called a continuous random sequence.

For an illustration, if X_n represents the temperature at the end of n^th hour of a day, then the states $\{X_n,\ 1 \le n \le 24\}$ can take any value in the interval and hence it is continuous.

If the state space E of a random process is discrete then the random process is called a discrete process or discrete state. A discrete random process is also called a chain.

For an illustration, if $X(t)$ represents the number of SMS received in a cellphone in the interval (0, t). Then $X(t)$ is a discreate random process with E={0,1,3,…}

If the state space E of a random process is continuous then the random process is called a continuous random process or continuous state process.

For an illustration, if $X(t)$ represents the maximum temperature recorded in a city in the interval (0, t). Then $X(t)$ is a continuous random process.

2.5. Markov Process

A Markov process is a random process with the memory loss property i.e., future of the process is depending only on the current state and not on the state in the past. Mathematically, a random process $\{X(t),\ t \epsilon T\}$ is said to be a Markov process if,

$$P\{X(t_{n+1}) \le x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0]$$

$$= P\{X(t_{n+1}) \le x_{n+1} \mid X(t_n) = x_n\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where, $t_0 < t_1 < \ldots < t_n < t_{n+1}$ and $\{x_0, x_1, \ldots, x_n\}$ are called the states of the process.

The above equation (1) states that the random process $X(t)$ at the time t_n is in the state x_n, the future state of the random process $X(t)$ at time t_{n+1} depends only on the present state x_n and not on the past states $x_{n-1}, x_{n-2}, \ldots, x_0$.

Examples of Markov process:

1. Any random process with independent increments
2. Board games played with dice, like Monopoly, Snakes and Ladder
3. Weather prediction model etc.

2.6. Markov Chains

A Markov process with discrete state is known as Markov chain. Thus, a discreate parameter Markov chain is defined as a set of random variables $\{X_n, n \geq 0\}$ with the Markov property i.e., given the present states the future and past states are independent. Mathematically it can be denoted as follows;

$$P\{X_{n+1} = y \mid x_n = x, x_{n-1} = x_{n-1}, \ldots, x_0 = x_0\} = P\{X_{n+1} = y|x_n = x\} \ldots\ldots \quad (2)$$

All possible values of $x_i\,(i = 1,2,\ldots)$ are form the state space.

Markov chain can be used:

1. To understand the market place for a product w.r.t. its competitive brands.
2. In Billing, Credit and Collection Procedure.
3. In Machine maintenance

2.7. Transition Probabilities

Let $\{X_n, n \geq 0\}$ be a Markov chain that takes on a finite or countable number of possible values without loss of generality. This set of possible values of the process is denoted by the set of non-negative integers $\{0, 1, 2, \ldots\}$.

If $X_n = i$, we say that the random process is in state 'i' at the time 'n'. The conditional probability $P\{X_{n+1} = j|X_n = i\}$ is called one step transition probability from state 'i' to state 'j' at the $(n+1)^{th}$ step and is denoted by $P_{ij}(n, n+1)$.

If one step transition probability is independent of 'n' i.e., $P_{ij}(n, n+1)$=$P_{ij}(m, m+1)$ the Markov chain in said to be a homogeneous Markov chain. Otherwise, the process is non-homogeneous Markov chain.

For a homogeneous Markov chain, the conditional probability becomes; $P\{X_n = j \mid x_o = i\}$. The above expression means the probability that the process is in state 'j' at step 'n' given that it was in state 'i' at the step '0'. This is called the n-step transition probability and is denoted by, $P_{ij}{}^{(n)} - P\{X_n - j \mid x_o - i\}$; where $P_{ij}{}^{(1)} = P_{ij}$ is the one step transition probability.

2.8. Transition Probability Matrix

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain with a discreate space $S = \{0, 1, 2, \ldots\}$. Then the one step transition probability from state i to state j is defined as;

$$P_{ij} = P\{X_{n+1} = j \mid x_n = i\} \text{ ; where i, j} \geq 0$$

Which is same for all the values of n as the Markov chain is homogeneous.

Now, the transition probability matrix of the process $\{X_n, n \geq 0\}$ is given by;

$$P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mn} \end{bmatrix}$$

Here P is the transition matrix and its elements satisfy, $P_{ij} \geq 0$, $\sum_{j=0}^{n} P_{ij} = 1$ for all $i = 0,1,2,\ldots,m$ and $j = 0,1,2,\ldots,n$.

Example: 4

Show that if P is a Markov matrix, then P^n is also a Markov matrix for any positive integer n.

Solution:

Let us consider a Markov matrix P of order m×m and a vector V of order m×1 as follows; $P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix}$ and V= $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

We shall prove this by method of induction. Let us check the above assertion for n=1.

Now, applying the conditions of TPM i.e., $P_{ij} \geq 0$, $\sum_{j=0}^{m} P_{ij} = 1$ for all $i = 0,1,2,\ldots,m$ and $j = 0,1,2,\ldots,m$ we have;

$$PV = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & \vdots & & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{j}^{m} P_{1j} \\ \sum_{j}^{m} P_{2j} \\ \vdots \\ \sum_{j}^{m} P_{mj} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = V$$

⇨ $PV = V$ ………………………………. (a)

The given statement is true for n=1.

Let us check the above assertion for n=2. Now, pre-multiplying both the sides of the above equation (a) we get;

$P(PV) = PV$

⇨ $P^2V = V$ (using equation 'a')

⇨ P^2 is a Markov matrix ……………………………………... (b)

Thus, the above statement is true for n=2.

Assume that P^k is a Markov matrix for some positive integer k i.e., $P^kV - V$ ……..……. (c)

Now, check the above statement is true for n=k+1. Now take;

$(P^{k+1}V) = P(P^kV)$

$= P(V)$ (using equation 'c')

= V (using equation 'a')

Thus, P^{k+1} is also a Markov matrix

…………………………………………………….. (d)

Hence, the above statement is true for any positive integer n.

2.9. Probability Distribution of a Markov Chain

Let $\{X_n, n > 0\}$ be a homogeneous Markov chain. Let $P_i(n)$ denote the probability that the process is in state 'i' in the nth step i.e., $P_i(n) = P\{X_n = i\}$ and $P_i(n) = [P_0(n), P_1(n), \ldots,]$; Where $\sum_{j=0}^{\infty} P_{ij} = 1$.

Then, $P_i(0) = P\{X_0 = i\}$; $i \geq 0$is called initial state probability and vector $P_i(0) = [P_0(0), P_1(0), \ldots,]$ is called initial state probability vector or initial probability distribution.

Similarly, $P_i(n) = [P_0(n), P_1(n), \ldots,]$ is called the state probability vector after the n steps or the probability distribution of the Markov chain.

2.10. Equilibrium Condition

One may observe that if no attempt is made to alter the transition probabilities, the system would eventually reach a point of equilibrium, i.e., for all practical purposes, no further changes would occur in the state of probabilities. After a Markov process has been in operation for a long time (many steps) a given state will tend to occur a fixed per cent of time. There is a limiting probability that the system in a finite (but large) number of transitions will reach steady steam equilibrium.

That is when n becomes very large each p_{ij} tends to fixed limits and each state probability vector $p^{(n)}$ approaches a constant value, i.e.,

$$p^{(n+1)} - p^{(n)} - p \text{ , independent of n.}$$

Thus, in the limiting case

$$\lim \quad p^{(n+1)} = \lim \text{ P } \quad p^{(n)} \text{ , becomes p=Pp}$$

Therefore, in the limit as ‘n’ approaches infinity, $p^{(n)}$becomes constant (i.e., independent) of time and the system is said to have reached a steady state equilibrium.

2.11. Algorithm of Markov Analysis

The following algorithm can be followed for a Markov analysis:

Step-1
- Using the problem data determine the probabilities associated with retention, gains and losses.

Step-2
- Then develope an appropriate state transition matrix.

Step-3
- Using the developed transition matrix, determine the expected future market shares for the next period as follows;
  [■8(market shares begining@of period 1)][■8(state transition@matrix)] = [■8(expected market shares@begining of period 2)]

Step-4
- If more than one expected market shares calculation is required then apply the following;
  [■8(Expected market shares @begining of period *k*)][■8(state transition@matrix)] = [■8(expected market shares@begining of period *k* +1)]

Step-5
- Using the state transition matrix developed in step-2 determine the equilibrium condition for the given problem.

Example: 5

Suppose there are two market products of brand A and B. Let each of these two brands have exactly 50% of the total market in same period and let the market be of a fixed size. The transition matrix is given below:

		To	
		A	B
From	A	0.9	0.1
	B	0.5	0.5

If the initial market shares breakdown is 50% for each brand, then determine their market shares in the steady state.

Solution:

Here the given state transition matrix is;

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

Using the above state transition matrix, we determine the expected market shares for the next period. We are given that the initial market shares for brand A and B are 50% each, which means the market shares at the beginning of the period 1 are $[0.5, \quad 0.5]$. After the promotional efforts applied to the brand A and B, the transition matrix indicates that in the second period, brand A will retain 90% of its customer and take away 50% of B's. Thus, the market share of A in the second period becomes;

$$0.5 \times 0.9 + 0.5 \times 0.5 = 0.7$$

and the corresponding market shares of B in the second period will be;

$$0.5 \times 0.1 + 0.5 \times 0.5 = 0.3$$

Continuing the algorithm,

$$\begin{bmatrix} \text{market shares begining} \\ \text{of period 1} \end{bmatrix} \begin{bmatrix} \text{state transition} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{expected market shares} \\ \text{begining of period 2} \end{bmatrix}$$

We have,

Market shares at the Beginning of period 1		State transition matrix		Expected market shares at the beginning of period 2
$[0.5 \quad 0.5]$	×	$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$	=	$[0.7 \quad 0.3]$

If the same transition matrix holds from one period to another, then the market shares of two brands for different periods are obtained as follows;

Market shares at the beginning of period 2		State transition matrix		Expected market shares at the beginning of period 3
$[0.7 \quad 0.3]$	×	$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$	=	$[0.78 \quad 0.22]$

Market shares at the		State transition matrix		Expected market shares at the beginning of period 4

beginning of period 3

$$[0.78 \quad 0.22] \times \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = [0.812 \quad 0.188]$$

Proceeding in the similar manner we can obtained the following table:

Period	Market shares of Brand 'A' (in %)	Market shares of Brand 'B' (in %)
0	50	50
1	70	30
2	78	22
3	81.2	18.8
4	82.48	17.52
5	82.992	17.008
6	83	17

The above table shows that, starting with 50%, 50% spent the market shares by A and B, after the 6 period of time the market shares changes to 83% and 17% respectively. Thus, the steady state market shares of A and B will be ${}^{5}/_{6}$ and ${}^{1}/_{6}$ respectively of the total market shares.

2.12. Chapman-Kolmogorov Equation

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain with transition probability matrix $P - [P_{ij}]$ and n-step transition probability matrix $P^{(n)} = [P_{ij}{}^{(n)}]$, Where $P_{ij}{}^{(n)} - P\{X_n - j \mid x_0 - i\}$ and $P_{ij}{}^{(1)} = P_{ij}$. Then the following properties hold:

(1) $P^{(n+m)} = P^{(n)} P^{(m)}$

(2) $P^{(n)} = P^n$ i.e., the n-step transition probability matrix is equal to the nth power of the one-step transition probability matrix P.

Example : 6

The transition probability matrix of a Markov chain $\{X_n, n \geq 0\}$ having three states 1,2 and 3 is given as follows;

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

The initial probability distribution is $P(0) = [0.5 \quad 0.3 \quad 0.2]$, then find the value$P(X_2 = 2)$.

Solution :

As $P(0) = [0.5 \quad 0.3 \quad 0.2]$ follows $P(X_0 = 1) = 0.5$, $P(X_0 = 2) = 0.3$ and $P(X_0 = 3) = 0.2$.

Applying Chapman-Kolmogorov equation we have;

$$P^{(2)} = P^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.48 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix}$$

Now, $P(X_2 = 2) = \sum_{i=1}^{3} P\{X_2 = 2 \mid X_0 = i\} \,.\, P\{X_0 = i\}$

$$= P_{12}^{(2)} P(X_0 = 1) + P_{22}^{(2)} P(X_3 = 2) + P_{32}^{(2)} P(X_0 = 3)$$

$$= 0.39 \times 0.5 + 0.48 \times 0.3 + 0.39 \times 0.2$$

$$= 0.195 + 0.144 + 0.078$$

$$= 0.417$$

2.13. Summary

In this section the decision making under different environment viz. certainty, uncertainty, conflict, risk etc. has been thoroughly discussed. In addition to this the procedure of Markov analysis involving different Markov models has also been discussed.

2.14. Key Terms

- Decision environment
- Decision making under certainty
- Decision making under conflict
- Decision making under uncertainty
- Decision making under risk
- Expected monetary value (EMV)
- Expected opportunity loss (EOL)
- Expected value of perfect information (EVPI)
- Markov analysis
- Markov models
- Markov process
- Markov Chains
- Transition probabilities

- Transition probability matrix
- Probability distribution of a Markov chain
- Equilibrium condition
- Algorithm of Markov analysis
- Chapman-Kolmogorov Equation

2.15 Check Your Progress

a. A payoff table for three different courses of action (A_1, A_2 and A_3) with 3 different states of nature (E_1, E_2 and E_3) with their respective probabilities (p) is given. Find out the best course of action.

Event	E_1	E_2	E_3
Probability → Acts ↓	0.2	0.5	0.3
A_1	2	1	-1
A_2	3	2	0
A_3	4	2	1

b. Given the following payoff table for 3 acts (A_1, A_2, A_3) and their events (E_1, E_2, E_3).

Action → State of Nature ↓	A_1	A_2	A_3
E_1	35	-10	-150
E_2	200	240	200
E3	550	640	750

The probability of the state of nature are 0.3, 0.4 and 0.3 respectively. Calculate the EMV and choose as the best action.

c. The demand for a seasonal product is given in a tabular form below;

Demand during the season	40	45	50	55	60	65
Probability	0.1	0.2	0.3	0.25	0.1	0.05

The product cost is 60 rupees per unit and sells at 80 rupees per unit. If the units are not sold in the season, then they will have no market value. Thus, a) Prepare a payoff and regret table, b) Find the expected payoffs and regret and c) Find the optimum act and EVPI.

d. An unbiased die is tossed repetitively. If $\square_{\square}$ denotes the maximum of the numbers occurring in the 1st 'n' trials, find the transition probability matrix p of the Markov chain $\{\square_{\square}\}$.Also find $\square^2$ and $\square(\square_2 = 0.6)$.

e. Suppose the probability of a dry day following a rainy day is $1/3$ and the probability of a rainy day following a dry day is $1/2$. Given that June 5 is a dry day, find the probability that June 8 is a dry day.

2.16 References

- Kalavathy, S (2013): Operations Research, 4th Edition, Vikas Publishing House, Pvt. Ltd.
- Murthy P. R. (2007): Operations Research, second edition, New Age International (P) Ltd., Publishers, ISBN (13): 978-81-224-2944-2.
- Swarup, K., Gupta, P. K., & Mohan, M. (2017): Operations Research, Sultan Chand & Sons, New Delhi.
- Gupta, R. K. (2010): Operations Research, Krishna Educational Publishers.
- Ravindran, Phillips and Solberg (2012): Operation Research: Principles and practice, Second edition, Wiley student edition.
- Taha, H. A. (2009): Operation research: An introduction, Eighth edition, Pearson Education Publication.
- Vohra, N. D.: Quantitative Management, Tata McGraw Hill
- Kapoor, V. K.: Operation Research, Sultan Chand & Sons
- Sharma, J. K.: Operation Research Theory & Applications, Macmillan India Limited

2.17 Model Questions

- The wholesaler of fruits promises delivery within four hours on all orders. The fruits are purchased on the previous day and delivered to the manager by 8.00 A.M. the next morning. Wholesaler’s daily demand for the fruits is given in the table below:

Fruits (in dozens)	7	8	9	10
Probability	0.1	0.2	0.4	0.3

  The wholesaler purchase fruits for 10 rupees per dozen and sells them for 30 rupees. All unsold fruits are donated to a nearby orphan. How many dozens of fruits should the wholesaler order each evening to maximize the profit of the fruit shop? What is the optimum expected profit?

- A company manufactures goods for a market in which the technology of the product is changing rapidly. The research and development department produced a new product which appears to have potential for commercial exploitation. A further 60000 rupees is required for development testing. The company has 100 customers and each customer might purchase at the most one unit of the product. Market research suggests that a selling price of 6000 rupees for each unit with total variable cost for manufacturing and selling estimate are 2000 rupees for each unit. As a result of previous experience of this type of market it has been possible to derive a probability distribution relating to the proportions of customers who will buy the product as follows:

Proportion of customers	0.04	0.08	0.12	0.16	0.20
Probability	0.1	0.1	0.2	0.4	0.2

Calculate the corrected opportunity losses given no other information than that stated above and state whether or not the company should develop the product.

- A bakery shopkeeper is faced with the problem of how many fruit cakes to buy in order to meet the day's demand. The shopkeeper prefers not to sell day old goods in competition with fresh products. The leftover fruit cakes are therefore a complete loss. On the other hand, if a customer desires a fruit cake and all of them have been sold, the disappointed customer will buy elsewhere and the sales will be lost. The shopkeeper has therefore collected information on the past sales on a selected 100-daysperiod as shown in the table below:

Sales per day	25	26	27	28
No of days	100	30	50	10
Probability	0.10	0.30	0.50	0.10

A fruit cake costs of 80 rupees and sells for 100 rupees. Construct the payoff table and the opportunity loss table. What is the optimum number of fruit cakes that should be bought each day? Also find the value of EVPI and interpret.

- The demand for the winter gloves in a winter season given as below:

Demand during the season	40	45	50	55	60	65
Probability	0.10	0.20	0.30	0.25	0.10	0.05

The product cost is 60 rupees per unit and it sells at 80 rupees per unit. If the units are not sold within the season, they will have no market value. Using the given information; i) Prepare a payoff and regret table, ii) Find the expected payoff and regret iii) Find the optimum act and EVPI.

- An automobile seller finds that the cost of holding per unit per week is 40 rupees and the cost of shortage per unit per week is 60 rupees. For one particular model of automobile, the probability distribution of weekly sale is as follows:

Weekly sales	0	1	2	3	4	5	6
Probability	0.10	0.10	0.20	0.25	0.15	0.15	0.05

How many units per week should the dealer order? Also find EVPI.

- A housewife buys 3 kinds of products say A, B and C. She never buys the same product in successive weeks. If she buys product A, then the next week she buys product B. However, if she buys B or C the next week, she is three times as likely to buy A as the other product. How often does she buy each of the 3 products?
- A person tosses a fair coin until 3 heads occur in a row. Let $\square_{\square}$ denotes the longest string of heads ending at n^{th} trial. Show that the process is Markovian. Find its transition matrix.

Unit-3
Game Theory

Structure:

3.1 Learning Objectives

After studying this chapter, you should be able to know

- understand how optimal strategies are formulated and used in conflict and competitive environment.
- understand the principles of two-person zero-sum games.
- apply dominance rules and compute value of the game using mixed strategies.
- apply minimax and maximin principle to compute the value of the game, when there is a saddle point.

3.2 Introduction

A game is a situation of conflict and competition in which two or more players are involved. The competitors are termed as players. A player may be an individual, individuals, or an organization. A few examples of competitive and conflicting decision environment are:

- Fixing the price of products, where the sale of any product is determined not only by its price but also by the price set by competitors for a similar product in the market.
- The success of any television programme mostly depends on what the competitors present in the same time slot and the programme they telecast.
- The success or failure of an advertising/marketing campaign depends on different types of services offered to the customers.

The theory of games provides mathematical models that may be useful where two or more competitors are involved under conditions of conflict and competition. However, such models provide an opportunity to a competitor to evaluate not only his personal decision alternatives (courses of action), but also the evaluation of the competitor's possible choices in order to win the game.

Game theory came into existence in 20th Century. John Von Neumann and Oscar Morgenstern published a book in 1944 named Theory of Games and Economic Behavior, in which they discussed how businesses of all types may use the techniques of game theory to determine the best strategies given a competitive business environment. The models in the theory of games can be classified based on the following factors:

- **Number of players:** If a game involves only two players (competitors), then it is called a **two-person game**. However, if the number of players are more, the game is referred to as **n-person game.**
- **Sum of gains and losses:** If the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a **zero-sum game**. Otherwise, it is said to be non-zero-sum game.
- **Strategy:** The strategy for a player is the list of all possible actions (moves, decision alternatives or courses of action) that are likely to be adopted by him for every **payoff** (outcome). It is assumed that the players are aware of the rules of the game governing their decision alternatives (or strategies). The outcome resulting from a particular strategy is also known to the players in advance and is expressed in terms of numerical values (e.g., money, per cent of market share or utility).

The particular strategy that optimizes a player's gains or losses, without knowing the competitor's strategies, is called **optimal strategy**. The expected outcome, when players use their optimal strategy, is called **value of the game**.

Generally, the following two types of strategies are adopted by players in a game:

(a)**Pure Strategy:** A particular strategy that a player chooses to play again and again regardless of other player's strategy, is referred as pure strategy. The objective of the players is to maximize their gains or minimize their losses.

(b)**Mixed Strategy**: A set of strategies that a player chooses on a particular move of the game with some fixed probabilities are called mixed strategies. Thus, there is a probabilistic situation and objective of each player is to maximize expected gain or to minimize expected loss by making the choice among pure strategies with fixed probabilities.

Mathematically, if p_j $(j = 1, 2, \ldots, n)$ is the probability associated with a pure strategy j to be chosen by a player at any point of time during the game, then the set S of n non-negative real numbers (probabilities) whose sum is unity associated with pure strategies of the player is written as:

$S = \{p_1, p_2, \ldots, p_n\}$ where $p_1 + p_2 + \ldots + p_n = 1$ and $p_j \geq 0$ of all j.

Remark: If a particular $p_j = 1$ $(j = 1, 2, \ldots, n)$ and all others are zero, the player is said to select pure strategy j.

3.3 Two-Person Zero-Sum Games

A game with two players, say A and B, is called a two-person zero-sum game, when one player's gain is equal to the loss of another player, so that total sum is zero.

Payoff matrix: The payoffs (a quantitative measure) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the payoff matrix.

Let player A have m strategies represented by: $A_1, A_2, \ldots, A_m$ and player B have n strategies represented by: $B_1, B_2, \ldots, B_n$. The numbers m and n need not be equal. The total number of possible outcomes is therefore $m \times n$. It is assumed that each player not only knows his own list of possible strategies but also of his competitor. For convenience, it is assumed that player A is always a gainer whereas player B a loser. Let a_{ij} be the payoff that player A gains from player B if player A chooses strategy i and player B chooses strategy j. Then the payoff matrix is shown in the Table

Player A's Strategies | Player B's Strategies B_1 B_2 $\cdots B_n$

$$\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Since player A is assumed to be the gainer, therefore he wishes to gain as large a payoff a_{ij} as possible, player B on the other hand would do his best to reach as small a value of a_{ij} as possible. Of course, the gain to player B and loss to A must be $-a_{ij}$.

Various methods discussed in this chapter to find value of the game under decision-making environment of certainty are as follows:

Assumptions of the game

1. Each player has a finite number of possible strategies (courses of action). The list may not be the same for each player.
2. Players act rationally and intelligently.
3. List of strategies of each player and the amount of gain or loss on an individual's choice of strategy is known to each player in advance.
4. One player attempts to maximize gains and the other attempts to minimize losses.
5. Both players make their decisions individually, prior to the play, without direct communication between them.
6. Both players select and announce their strategies simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
7. The payoff is fixed and determined in advance.

3.4 Pure Strategies (Minimax and Maximin Principles): Games with Saddle Point

By convention, the payoff table for the player whose strategies are represented by rows (say player A) is constructed. The columns represent the returns for the player B. The players must select their respective strategies so that they are able to optimize their payoffs. Such a decision-making criterion is called the minimax-maximin principle.

- **Maximin principle:** For player A the minimum value in each row represents the least gain (payoff) to him, if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game.
- **Minimax principle:** For player B, who is assumed to be the looser, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategy. These are written in the payoff matrix by column maxima. He will then select the strategy that gives the minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game.
- **Optimal strategy:** A course of action that puts any player in the most preferred position, irrespective of the course of action his competitor(s) adopt, is called as optimal strategy. In other words, if the maximin value equals the minimax value, then the game is said to have a saddle (equilibrium) point and the corresponding strategies are called optimal strategies.
- **Value of the game:** This is the expected payoff at the end of the game, when each player uses his optimal strategy, i.e., the amount of payoff, V, at an equilibrium point. A game may have more than one saddle points. A game with no saddle point is solved by choosing strategies with fixed probabilities.

Remarks:

1. The value of the game, in general, satisfies the equation: maximin value $\leq$ V $\leq$ minimax value.
2. A game is said to be a fair game if the lower (maximin) and upper (minimax) values of the game are equal and both equals zero.
3. A game is said to be strictly determinable if the lower (maximin) and upper (minimax) values of the game are equal and both equal the value of the game.

Rules to Determine Saddle Point

The reader is advised to follow the following three steps, in this order, to determine the saddle point in the payoff matrix.

1. Select the minimum (lowest) element in each row of the payoff matrix and write them under 'row minima' heading. Then, select the largest element among these elements and enclose it in a rectangle,

2. Select the maximum (largest) element in each column of the payoff matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle,
3. Find out the element(s) that is same in the circle as the well as rectangle and mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or equilibrium) point.

Example 1: For the game with payoff matrix:

Player B

Player A B_1 B_2 B_3

$$\begin{matrix} A_1 \\ A_2 \end{matrix} \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

determine the optimal strategies for players A and B. Also determine the value of game. Is this game (i) fair? (ii) strictly determinable?

Solution: In this example, gains to player A or losses to player B are represented by the positive quantities, whereas, losses to A and gains to B are represented by negative quantities. It is assumed that A wants to maximize his minimum gains from B. Since the payoffs given in the matrix are what A receives, therefore, he is concerned with the quantities that represent the row minimums. Now A can do no worse than receive one of these values. The best of these values occurs when he chooses strategy A_1. This choice provides a payoff of – 2 to A when B chooses strategy B_3. This refers to A's choice of A_1 as his maximum payoff strategy because this row contains the maximum of A's minimum possible payoffs from his competitor B.

[*Maximin principle maximizes the player's minimum gains.*

Value of the game is the expected gain or loss in a game when a game is played a large number of times.

Saddle point is the payoff value that represents both minimax and maximin value of the game.]

Player B

Player A B_1 B_2 B_3 Row Minimum

$$\begin{matrix} A_1 \\ A_2 \end{matrix} \begin{bmatrix} 1 & 2 & -2 \\ 6 & 4 - & 6 \end{bmatrix} \begin{matrix} -2 \\ -6 \end{matrix}$$ Maximin

Column Maximum	6	4	-2	Minimax

Similarly, it is assumed that B wants to minimize his losses and wishes that his losses to A be as small as possible. The column maximums also represent the greatest payments B might have to make to A. The smallest of these losses is – 2, which occurs when A chooses his course of action, A_1 and B chooses his course of action, B_3. This choice of B_3 by B is his minimax loss strategy because the amount of this column is the minimum of the maximum possible losses.

This value is referred to as saddle point. The payoff amount in the saddle-point position is also called value of the game. For this game, value of the game is, V = – 2, for player A. The value of game is always expressed from the point of view of the player whose strategies are listed in the rows. The game is strictly determinable. Also, since the value of the game is not zero, the game is not fair.

Example 2: A company management and the labour union are negotiating a new three-year settlement. Each of these has 4 strategies:

I: Hard and aggressive bargaining II: Reasoning and logical approach

III: Legalistic strategy IV: Conciliatory approach

The costs to the company are given for every pair of strategy choice.

Company Strategies

Union Strategies	I	II	III	IV
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	−5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution: Applying the rule of finding out the saddle point, we obtain the saddle point that is enclosed both in a circle and a rectangle, as shown in Table

Company Strategies

Union Strategies	I	II	III	IV	Row minimum	
I	20	15	12	35	12	← Maximin
II	25	14	8	10	8	
III	40	2	10	5	2	
IV	−5	4	11	0	−5	
Column maximum	40	15	12	35		

↑ Minimax

Since Maximin = Minimax = Value of game = 12, therefore the company will always adopt strategy III – Legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

Example 3: Find the range of values of p and q that will render the entry (2, 2) a saddle point for the game:

Player B

Player A B_1 $B_2 B_3$

$$\begin{matrix} A_1 \\ B_1 \\ C_1 \end{matrix} \begin{bmatrix} 24 & 5 & 2 \\ 10 & 7 & q \\ 4 & p & 6 \end{bmatrix}$$

Solution: Let us ignore the values of p and q in the payoff matrix, and then determine the maximin and minimax values in the usual manner, as shown in Table:

Player B

Player A $B_1 B_2$ B_3 Row minimum

$$\begin{matrix} A_1 \\ B_1 \\ C_1 \end{matrix} \begin{bmatrix} 24 & 5 & 2 \\ 10 & 7 & q \\ 4 & p & 6 \end{bmatrix} \begin{matrix} 2 \\ 7 \\ 4 \end{matrix} \quad \leftarrow \text{Maximin}$$

Column maximum 10 7 6

↑ Minimax

As shown in Table 4, since there exists no unique saddle point, therefore, the saddle point will exist at the position (2, 2) only when $p \leq 7$ and $q > 7$.

Example4: For what value of λ, the game with following pay-off matrix is strictly determinable?

Player B

Player A $B_1 B_2 B_3$

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} -\lambda & 6 & -2 \\ -1 & \lambda & 4 \\ -2 & 4 & -\lambda \end{bmatrix}$$

Solution First, ignoring the value of λ, determine the maximin and minimax values of the pay-off matrix, as shown below:

Player B

Player A B_1 $B_2 B_3$ Row minimum

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} -\lambda & 6 & -2 \\ -1 & \lambda & 4 \\ -2 & 4 & -\lambda \end{bmatrix} \begin{matrix} -2 \\ -7 \\ -2 \end{matrix} \quad \leftarrow \text{Maximin}$$

Column maximum –1 6 –2

↑ Minimax

Since saddle point in the above table is not unique, the value of the game lies between –1 and 2, i.e. $-1 \leq V \leq 2$. For strictly determinable game, we must have $-1 \leq \lambda \leq 2$.

3.5 Mixed Strategies: Game Without Saddle Point

In certain games, no saddle point exists, i.e., maximin value ≠ minimax value. In all such cases, players must choose the mixture of strategies to find the value of game and an optimal strategy.

The value of game obtained by the use of mixed strategies represents the least payoff, which player A can expect to win and the least which player B can expect to lose. The expected payoff to a player in a game with payoff matrix $[a_{ij}]$ of order m × n is defined as:

$E(p, q) = \Sigma p_i a_{ij} q_j = P^T A Q$ (in matrix notation), i = 1,2, …, m; j = 1, 2,… n

Where, $P = (p_1, p_2, \ldots, p_m)$ and $Q = (q_1, q_2, \ldots, q_n)$ denote probabilities (or relative frequency with which a strategy is chosen from the list of strategies) associated with m strategies of player A and n strategies of player, B respectively,

where $p_1 + p_2 + \ldots + p_m = 1$ and $q_1 + q_2 + \ldots + q_n = 1$.

A mixed strategy game can be solved by using following methods:

• Algebraic method • Analytical or calculus method

• Matrix method • Graphical method, and • Linear programming method.

Remark For solving a 2 × 2 game, without a saddle point, the following formula is also used. If payoff matrix for player A is given by:

$$\text{Player } A \begin{array}{c} \text{Player } B \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

then the following formulae are used to find the value of game and optimal strategies:

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}; \quad q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

where $p_2 = 1 - p_1; \quad q_2 = 1 - q_1$

and $$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

3.6 The Rules (Principles) of Dominance

The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix that are inferior (less attractive) to at least one of the remaining rows and/or columns (strategies), in terms of payoffs to both the players. Rows and/or columns once deleted can never be used for determining the optimum strategy for both the players.

The rules of dominance are especially used for the evaluation of two-person zero-sum games without a saddle (equilibrium) point. Certain dominance principles are stated as follows:

1. For player B, who is assumed to be the loser, if each element in a column, say Cr is greater than or equal to the corresponding element in another column, say Cs in the payoff matrix, then the column Cr is said to be dominated by column Cs and therefore, column Cr can be deleted from the payoff matrix. In other words, player B will never use the strategy that corresponds to column Cr because he will lose more by choosing such strategy.

2. For player A, who is assumed to be the gainer, if each element in a row, say Rr, is less than or equal to the corresponding element in another row, say Rs, in the payoff matrix, then the row Rr is said to be dominated by row Rs and therefore, row Rr can be deleted from the payoff matrix. In other words, player A will never use the strategy corresponding to row Rr, because he will gain less by choosing such a strategy.

3. A strategy say, k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination would be decided as per rules 1 and 2 above.

Remark Rules (principles) of dominance discussed are used when the payoff matrix is a profit matrix for the player A and a loss matrix for player B. Otherwise the principle gets reversed.

Example 5: Players A and B play a game in which each has three coins, a 5p, 10p and a 20p. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, then A wins B's coin. But, if the sum is even, then B wins A's coin. Find the best strategy for each player and the values of the game.

Dominance rules is the procedure to reduce the size of the payoff matrix according to the tendency of players.

Solution The payoff matrix for player A is

Player B

Player A5p: $B_1$10p: $B_2$20p: B_3

$$\begin{matrix} 15p: A1 \\ 10p: A2 \\ 20p: A3 \end{matrix} \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & 20 & -20 \end{bmatrix}$$

It is clear that this game has no saddle point. Therefore, further we must try to reduce the size of the given payoff matrix as further as possible. Note that every element of column B3 (strategy B3 for player B) is more than or equal to every corresponding element of row B2 (strategy B2 for player B). Evidently, the choice of strategy B3, by the player B, will always result in more losses as compared to that of selecting the strategy B2. Thus, strategy B3 is inferior to B2. Hence, delete the B3 strategy from the payoff matrix. The reduced payoff matrix is shown below:

Player B

Player A B_1 $B_2 B_3$

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & 20 & -20 \end{bmatrix}$$

After column B_3 is deleted, it may be noted that strategy A_2 of player A is dominated by his A_3 strategy, since the profit due to strategy A_2 is greater than or equal to the profit due to strategy A_3, regardless of which strategy player B selects. Hence, strategy A_3 (row 3) can be deleted from further consideration. Thus, the reduced payoff matrix becomes:

Player B

Player A$B_1 B_2$ Row minimum

$$\begin{matrix} A_1 \\ A_2 \end{matrix} \begin{bmatrix} -5 & 5 \\ 5 & -10 \end{bmatrix} \begin{matrix} -5 \\ -10 \end{matrix} \quad \leftarrow \text{Maximin}$$

Column maximum 5 –10

↑ Minimax

As shown in the reduced 2 × 2 matrix, **the maximin value is not equal to the minimax value**. Hence, there is no saddle point and one cannot determine the point of equilibrium. For this type of game situation, it is possible to obtain a solution by applying the concept of mixed strategies.

3.7 Games without Saddle Point

Let player A have m pure strategies and player B have n pure strategies with corresponding probabilities as given in the table below:

Player B

Player A B_1 B_2 ... B_n **Probability**

$$\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{matrix}$$

Probability q_1 q_2 ... q_n

$a_{11}\, p_1 + a_{21}\, p_2 + \ldots + a_{m1}\, p_m \geq V$

$a_{12}\, p_1 + a_{22}\, p_2 + \ldots + a_{m2}\, p_m \geq V$

……. (1)

$a_{1n}\, p_1 + a_{2n}\, p_2 + \ldots + a_{mn}\, p_m \geq V$

where, $p_1 + p_2 + \ldots + p_m = 1$ and $p_i \geq 0$ for all i

Similarly, the expected loss to player B, when player A selects strategies $A_1, A_2, \ldots, A_m$, one by one, can also be determined. Since player B is the loser player, therefore, he must have:

$a_{11}\, q_{1m} + a_{12}\, q_{2m} + \ldots + a_{1n} q_{nm} \leq V$

$a_{21}\, q_{1m} + a_{22}\, q_{2m} + \ldots + a_{2n} q_{nm} \leq V$

…. (2)

$a_{m1}\, q_{11} + a_{m2}\, q_2 + \ldots + a_{mn} q_{nm} \leq V$

where, $q_1 + q_2 + \ldots + q_n = 1$ and $q_j \geq 0$ for all j.

To get the values of p_i's and q_j's, the above inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations, so obtained, is inconsistent, then at least one of the inequalities must hold as a strict inequality. The solution can now be obtained only by applying the trial-and-error method.

Example 6: In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and losses 1/2 unit of value when there is one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game to A.

Solution: The payoff matrix for the given matching coin games is given by:

Player B

Player A $B_1 B_2$

$$\begin{matrix} A_1 \\ A_2 \end{matrix} \begin{bmatrix} -2/1 & -1/2 \\ -1/2 & -2/0 \end{bmatrix}$$

As the payoff matrix does not have a saddle point, the game will be solved by the algebraic method. For Player A let p_1 and p_2 be probabilities of selecting strategy A_1 and

A_2, respectively. Then the expected gain to player A, when player B uses its B_1 and B_2 strategies, respectively, is given by:

$p_1 - (1/2)\, p_2 \geq V$; B selects B_1 strategy (1)

$-(1/2)\, p_2 + 0.\, p_2 \geq V$; B selects B_2 strategy (2)

where, $p_1 + p_2 = 1$ (3)

For obtaining value of p_1 and p_2, we consider inequalities (1) and (2) as equations and then with the help of Eq. (3), we get $p_1 = -2V$ and $p_2 = -6V$.

Substituting these values of p_1 and p_2 in Eq. (3) we get $V = -1/8$. Thus, $p_1 = 0.25$ and $p_2 = 0.75$.

For Player B Let q_1 and q_2 be the probabilities of selecting strategies B_1 and B_2, respectively. Then the expected loss to player B, when player A uses its A_1 and A_2 strategies, respectively is given by:

$q_1 - (1/2).\, q_2 \leq V$; A selects A_1 strategy (4)

$-(1/2).\, q_1 + 0.q_2 \leq V$; A selects A_2 strategy (5)

and $q_1 + q_2 = 1$ (6)

Consider inequalities (4) and (5) as equations and then, with the help of Eq. (6), we get $q_1 = 2V$ and $q_2 = -6V$.

Substituting values of q_1 and q_2 in Eq. (6), we get $V = -1/8$. Thus, $q_1 = 0.25$ and $q_2 = 0.75$.

Hence, the probability of selecting strategies optimally for players A and B are (0.25, 0.75) and (0.25, 0.75), respectively, and the value of the game is **V = – 1/8.**

Example 7: Solve the game whose payoff matrix is given below:

Player B

Player A $B_1 B_2\ B_3\ B_4$

$$\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$$

Solution: It is clear that this game has no saddle point. Therefore, we try to reduce the size of the given payoff matrix by using dominance principles. From player A's point of view, the first row is dominated by the third row, yielding the reduced 3 × 4 payoff matrix. In the reduced matrix from player B's point of view, the first column is dominated by the third column. Thus, by deleting the first row and then the first column, the reduced payoff matrix so obtained is:

Player B

Player A $B_2\ B_3\ B_4$

$$\begin{matrix} A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

Now it may be noted that none of the pure strategies of players A and B is inferior to any of their other strategies. However, the average of payoffs due to strategies B_3 and B_4, {(2 + 4)/2; (4 + 0)/2; (0 + 8)/2} = (3, 2, 4) is superior to the payoff due to strategy B_2 of player B. Thus, strategy B_2 may be deleted from the matrix. The new matrix so obtained is:

Player B

Player A $B_3 B_4$

$$\begin{matrix} A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ 0 & 8 \end{bmatrix}$$

Again, in the reduced matrix, the average of the payoffs due to strategies A_3 and A_4 of player A, i.e. {(4 + 0)/2; (0 + 8)/2} = (2, 4) is the same as the payoff due to strategy A_2. Therefore, player A will gain the same amount even if the strategy A_2 is never used. Hence, after deleting the strategy A_2 from the reduced matrix the following new reduced 2 × 2 payoff is obtained:

Player B

Player A$B_3 B_4$

$$\begin{matrix} A_3 \\ A_4 \end{matrix} \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$

This game has no saddle point. Let player A choose his strategies A_3 and A_4 with probability p_1 and p_2, respectively, such that $p_1 + p_2 = 1$. Also let player B choose his strategies with probability q_1 and q_2, respectively, such that $q_1 + q_2 = 1$. Since both players want to retain their interests unchanged, therefore,

we may write:

$4p_1 + 0.p_2 = 0.p_1 + 8p_2$ $4q_1 + 0.q_2 = 0.q_1 + 8q_2$

$4p_1 = 8(1 – p_1)$ $4q_1 = 8 (1 – q_1)$

$4p_1 = 2/3$ $4q_1 = 2/3$

We find that the optimal strategies of player A and player B in the original game are (0, 0, 2/3, 1/3) and (0, 0, 2/3, 1/3), respectively. The value of the game can be obtained by putting value of p1 or q1 in either of the expected payoff equations above. That is:

Expected gain to A	Expected loss to B
$4p_1 + 0.p_2 = 4(2/3) = 8/3$	$4q_1 + 0q_2 = 4(2/3) = 8/3$

3.8 Graphical Solution of Games

The graphical method very is useful for the game where the payoff matrix is of the size 2 × n or m × 2, i.e., the game with mixed strategies that has only two un-dominated pure strategies for one of the players in the two-person zero-sum game.

Optimal strategies for both the players have non-zero probabilities for the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be applied.

Consider the following 2 × n payoff matrix of a game, without saddle point.

Player B

Player A B_1 B_2...B_n Probability

A_1 a_{11} a_{12} ...a_{1n} p_1

A_2 a_{21} a_{22} ... a_{2n} p_2

Probability q_1 q_2... q_n

Player A has two strategies A_1 and A_2 with probability of their selection p_1 and p_2, respectively, such that $p_1 + p_2 = 1$ and $p_1, p_2 \geq 0$. Now for each of the pure strategies available to player B, the expected pay off for player A would be as follows:

B's Pure Strategies A's Expected Payoff

B_1 $a_{11}p_1 + a_{21} p_2$

B_2 $a_{12} p_1 + a_{22} p_2$

……………

B_n $a_{1n}p_1 + a_{2n} p_2$

According to the maximin criterion for mixed strategy games, player A should select the value of probability p1 and p2 so as to maximize his minimum expected payoffs. This may be done by plotting the straight lines representing player A's expected payoff values.

The highest point on the lower boundary of these lines will give the maximum expected payoff among the minimum expected payoffs and the optimum value of probability p_1 and p_2.

Now, the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. This helps in reducing the size of the game to (2 × 2), which can be easily solved by any of the methods discussed earlier.

The (m × 2) games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lowest point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q_1 and q_2.

Example 8: Use the graphical method for solving the following game and find the value of the game.

Player B

Player A B_1 B_2 B_3 B_4

$A_1$2 2 3 –2
$A_2$432 6

Solution The game does not have a saddle point. If the probability of player A's playing A_1 and A_2 in the strategy mixture is denoted by p_1 and p_2, respectively, where $p_2 = 1 - p_1$, then the expected payoff (gain) to player A will be

B's Pure Strategies A's Expected Payoff

$B_1$2 p_1 + 4 p_2
B_2 2 p_1 + 3 p_2
$B_3$3 p_1 + 2 p_2
B_4– 2 p_1 + 6 p_2
These four expected payoff lines can be plotted on a graph to solve the game.

A graphic solution is shown in Fig.1. Here, the probability of player A's playing A_1, i.e., p_1 is measured on the x-axis. Since p_1 cannot exceed 1, the x-axis is cutoff at $p_1 = 1$. The expected payoff of player A is measured along y-axis. From the game matrix, if player B plays B_1, the expected payoff of player A is 2 when A plays $A_1$with $p_1 = 1$ and 4 when A plays A_2 with p1 = 0. These two extreme points are connected by a straight line, which shows the expected payoff of A when B plays B_1. Three other straight lines are similarly drawn for B_2, B_3 and B_4.

It is assumed that player B will always play his best possible strategies yielding the worst result to player A. Thus, the payoffs (gains) to A are represented by the lower boundary when he is faced with the most unfavourable situation in the game.

Since player A must choose his best possible strategies in order to realize a maximum expected gain, the highest expected gain is found at point P, where the two straight lines

$E_3 = -3p_1 + 2p_2 = -3p_1 + 2(1 - p_1)$
$E_4 = -2p_1 + 6p_2 = -2p_1 + 6(1 - p_1)$,

meet. In this manner the solution to the original (2 × 4) game reduces to that of the game with payoff matrix of size (2 × 2) as given below:

Player B

Player AB_3B_4

A_1 3 –2
$A_2$2 – 6

The optimum payoff to player A can now be obtained by setting E_3 and E_4 equal and solving for p_1, i.e.

$3p_1 + 2(1 - p_1) = -2p_1 + 6(1 - p_1)$ or $p_1 = 4/9$; $p_2 = 1 - p_1 = 5/9$

Substituting the value of p_1 and p_2 in the equation for E_3 (or E_4) we have:

Value of the game, **V** = (3 × 4/9) +(2 × 5/9) = **22/9**

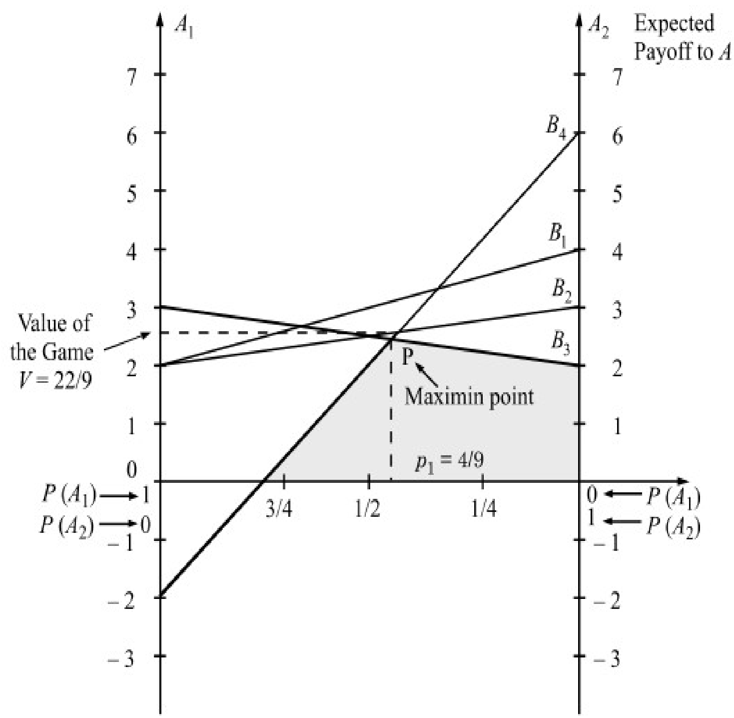
The optimal strategy mix of player B can also be found in the same manner as for player A. If the probabilities of B's selecting strategy B_3 and B_4 are denoted by q_3 and q_4, respectively, then the expected loss to B will be:

$L_3 = 3q_3 - 2q_4 = 3q_3 - 2(1 - q_3)$ (if A selects A_1)
$L_4 = 2q_3 + 6q_4 = 2q_3 + 6(1 - q_3)$ (if A selects A_2)

To solve for q3, equate the two equations:

$3q_3 - 2(1 - q_3) = 2q_3 + 6(1 - q_3)$ or $q_3 = 8/9$; $q_4 = 1 - q_3 = 1/9$

Fig. 1



Substituting the value of q_3 and q_4 in the equation for L_3 (or L_4), we have

Value of the game, **V** = 3 × 8/9 – 2 × 1/9 = **22/9.**

Example 9: Obtain the optimal strategies for both persons and the value of the game for two-person zero-sum game whose payoff matrix is as follows:

Player B

Player A $B_1 B_2$

A_1 1 –3
A_2 3 5
A_3 –1 6
A_4 4 1
A_5 2 2
A_6 –5 0

Solution The game does not have any saddle point. If the probability of player B's playing strategies B_1 and B_2 in the strategy mix is denoted by q_1 and q_2 such that $q_1 + q_2 = 1$, then the expected payoff to player B will be:

A's Pure Strategies B's Expected Payoff

$A_1 q_1 – 3q_2$

$A_2 3q_1 + 5q_2$

$A_3 – q_1 + 6q_2$

$A_4 4q_1 + q_2$

$A_5\ 2q_1 + 2q_2$

$A_6 – 5q_1 + 0q_2$

The six expected payoff lines can be plotted on the graph to solve the game. A graphic solution is shown in Fig.2 where the probability of player B's playing B1, i.e., q_1 is measured on the x-axis. Since q1 cannot exceed 1, therefore the x-axis is cutoff at $q_1 = 1$. The expected payoff of player B is measured along y-axis. From the game matrix, if player A plays A_1, the expected payoff of player B is 1 when he plays B_1 with $q_1 = 1$ and –3 when he plays B_2 with $q_1 = 0$. These two extreme points are connected by a straight line, which shows the expected payoff to B when A plays A_1. Five other straight lines are similarly drawn for A_2 to A_6.

It is assumed that player A will always play his best possible strategies, yielding the worst result to player B. Thus, payoffs (losses) to B are represented by the upper boundary when he is faced with the most unfavourable situation in the game. According to the minimax criterion, player B will always select a combination of strategies B_1 and B_2, so that he minimizes the losses. Even in this case the optimum solution occurs at the intersection of the two payoff lines.

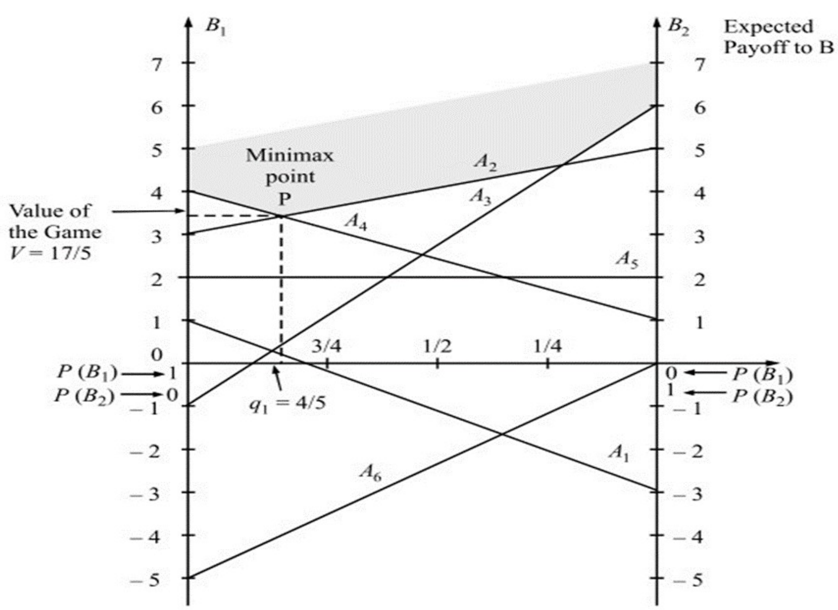


Fig.2

$E_3 = 3q_1 + 5q_2 = 3q_1 + 5(1 - q_1)$

$E_4 = 4q_1 + 4q_2 = 4q_1 + (l - q_1)$

The solution to the original (6 × 2) game reduces to that of the game with payoff matrix of size

(2 × 2) as shown below:

Player B

Player A$B_1 B_2$

$A_2$3 5

A_4 4 1

Now using the usual method of solution for a (2 × 2) game, the optimum strategies can be obtained as given below:

Player A: (0, 3/5, 0, 2/5, 0, 0); Player B: (4/5, 1/5) and, Value of the game, **V = 17/5**

Unit-4
Simulation

Structure :

4.1 Learning Objectives

After studying this chapter, you should be able to know

- distinguish between analytical and simulation models.
- learn the importance of simulation modelling.
- understand the advantages and disadvantages of simulation.
- use Monte Carlo simulation technique for solving various types of problems.
- develop random number intervals and use them

4.2 Introduction

Simulation that is not an optimizing technique, helps decision-makers to perform experiment with new values of variables and/ or parameters. The following few examples illustrate scope of applications of simulation.

1. Aircraft design engineers use wind tunnels to simulate the effect of air turbulence on various parts of an airplane before finalizing its design.
2. Aircraft pilots or Astronauts are trained in a simulator to expose them with various situations that they are likely to face in the sky while flying real aircraft or spacecraft.
3. Hospital management simulate alternative schedules for the ambulances, their locations, the response time to an emergency call and, the overall service quality and the costs incurred.

4. Computer designers simulate with a computer system configuration (such as speed, size, computing qualities, memory) in terms of costs.
5. Managers simulate alternative work schedules and use of new manufacturing technologies (such as Just-in-Time manufacturing, flexible manufacturing, etc.) that would give greater output in the productivity at a relatively small cost.
6. A queuing system decision-makers simulates the effect of probabilistic nature of arrival rate of customers and the service rate of the server to serve the customer better.

Other problems, such as location of bank branches, the deployment of fire stations, routing and dispatching when roads are not secured, the location and the utilization of recreational facilities (such as parks, public swimming pools, etc.) and many other problems could be studied through simulation. Simulation is also widely used in military practices and operations.

4.3 Simulation Defined

Simulation is one of the operations research techniques used for representing real-life problems through numbers and mathematical symbols that can be readily manipulated. For example, games such as chess to simulate battles, backgammon to simulate racing, and other games to simulate hunting and diplomacy were already invented where decision-makers used simulation to gain the ability to experiment with a situation under controlled conditions.

Today, a modern game like monopoly simulates the competitive arena of real estate. Many have played baseball with a deck of cards which has hits, strikeouts and walks with a cardboard, diamond and plastic chips as runners. The distribution of hits, runs and outs, etc., in a deck of cards serves as a realistic reflection of the overall average with which each would occur in real life. Now the availability of computer software makes it possible to deal with large quantity of details that can be incorporated into a model and also the ability to conduct many 'experiments' (i.e., replicating all the possibilities). Mathematician Von Neumann and Stan Ulam, in the late 1940s, developed the term Monte Carlo analysis while trying first to 'break' the Casino at Monte Carlo and subsequently, applying it to the solution of nuclear shielding problems that were either too expensive for physical experimentation, or too complicated for treatment by the already known mathematical techniques.

Few definitions of simulation are stated below:

*A simulation of a system or an organism is the operation of a model or simulator which is a representation of the system or organism. The model is amenable to manipulation which would be impossible, too expensive or unpractical to perform on the entity it portrays. The operation of the model can be studied and for it, properties concerning the behaviour of the actual system can be inferred. — Shubik

* Simulation is the process of designing a model of a real system and conducting experiments with this model for the purpose of understanding the behaviour (within the limits imposed by a criterion or set of criteria) for the operation of the system. — Shannon

* Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real-world system over extended periods of time.

— Naylor et al.

These definitions pointed out that simulation can be equally applied to military war games, business games, economic models, etc. Also, simulation involves logical and mathematical modelling that involves the use of computers to test the behaviour of a system using iterations or successive trials under realistic conditions.

For operations research practitioners, simulation is a problem-solving technique that uses a computer-aided experimental approach to study problems which otherwise is not possible through analytical methods.

4.4 Types of Simulation

There are several types of simulation. A few of them are listed below:

1. Deterministic versus probabilistic simulation: The deterministic simulation involves cases in which a specific outcome is certain for a given set of inputs. Whereas probabilistic simulation deals with cases that involves random variables and obviously the outcome cannot be known with certainty for a given set of inputs.

2. Time dependent versus time independent simulation: In time independent simulation it is not important to known exactly when the event is likely to occur. For example, in an inventory control situation, even if decision-maker knows that the demand is three units per day, but it is not necessary to know when demand is likely to occur during the day. On the other hand, in time dependent simulation it is important to know the exact time when the event is likely to occur. For example, in a queuing situation the exact time of arrival should be known (to know that the customer will have to wait).

3. Interactive simulation: Interactive simulation uses computer graphic displays to present the consequences of change in the value of input variation in the model. The decisions are implemented interactively while the simulation is running. These simulations can show dynamic systems that evolve over time in terms of animation. The decision-maker watches the progress of the simulation in an animated form on a graphics terminal and can alter the simulation as it progresses.

4. Business games: Business game simulation model involves several participants who need to play a role in a game that simulates a realistic competitive situation. Individuals or teams compete to achieve their goals, such as profit maximization, in

competition or cooperation, with the other individuals or teams. The few advantages of business games are:

(i) participants learn much faster and the knowledge and experience gained are more memorable than passive instruction.

(ii) complexities, inter functional dependencies, unexpected events, and other such factors can be introduced into the game for evoking special circumstances.

(iii) the time compression – allowing many years of experience in only minutes or hours – lets the participants try out actions that they would not be willing to risk in an actual situation and see the result in the future.

(iv) provide insight into the behaviour of an organization. The dynamics of team decision-making style highlight the roles assumed by individuals on the teams, the effect of personality types and managerial styles, the emergence of team conflict and cooperation, and so on.

5. Corporate and financial simulations: The corporate and financial simulation is used in corporate planning, especially the financial aspects. The models integrate production, finance, marketing, and possibly other functions, into one model either deterministic or probabilistic when risk analysis is desired.

4.5 Steps of Simulation Process

The process of simulating a system consists the following steps:

1. Defining the problem: Define the scope of study and the level of details that is required to derive desired results. Thus decision-makers should have clear idea about what is to be accomplished. For example, if a simulation study is to be done for arrival patterns of customers in a queuing system, then scope of study should decide certain hours of the day.

2. Identifying the decision variables and setting performance criterion Once problem is defined, the next step is to understand objectives for using simulation and degree (extent) with which these objectives shall be measured. In other words, before the start of simulation study, the decision-maker must ascertain how a system will behave given a set of input variables (conditions). For example, in an inventory control situation, the demand (consumption rate), lead time and safety stock are identified as decision variables. These variables shall be responsible to measure the performance of the system in terms of the total inventory cost under the decision rule – when to order.

3. Developing a simulation model: For developing a simulation model, an understanding of the relationships among the elements of the system being studied is required. For this purpose, the influence diagram (drawn in a variety of different ways) is useful. This is because simulation models for each of these diagrams may be formulated until one seems better or more appropriate than the other. Even after one has been chosen, it may be modified again and again before an acceptable version is arrived at.

4. Testing and validating the model: The purpose of validation is to check whether the model adequately reflects performance of real system. This requires comparing a model with the actual system – a validation process. A validated model should behave similar to the system under study. Discrepancies (if any) should be rectified in order to achieve objectives of simulation. The validation process requires

   (i) determining whether the model is internally correct in a logical and programming sense called internal validity and

   (ii) determining whether it represents the system under study called external validity.

   The first step involves checking the equations and procedures in the model for accuracy, both in terms of mistakes (or errors) and in terms of properly representing the system under study. After verifying internal validity, the model is tested by putting different values to variables into the model and observing whether it replicates what happens in reality. The decision-maker can make changes in the assumptions or input data and see the effect on the outputs. If the model passes this test, extreme values of the input variables are entered and the model is checked for the expected output.

5. Designing of the experiment Experimental design refers to controlling the conditions of the study such as the variables to be included and recording the effect on the output. The design of experiment requires determining

   (i) the parameters and variable in the model,

   (ii) levels of the parameters to use,

   (iii) the criterion to measure performance of the system,

   (iv) number of times the model will be replicated,

   (v) the length of time of each replication, and so on.

For example, in a queuing simulation we may consider the arrival and service rates to be constant but the number of servers and the customers waiting time may vary (dependent variable).

6. Run the simulation model Run the model using suitable computer software to get the results in the form of operating characteristics.

7. Examine the outputs Examine the outputs of the experiments and their reliability. If the simulation process is complete, then select the best course of action (or alternative), otherwise make desired changes in model decision variables, parameters or design, and return to Step 3.

4.6 Advantages and Disadvantages of Simulation

Advantages :

1. This approach is suitable to analyze large and complex real-life problems that cannot be solved by the analytical methods.

2. It facilitates to study the interactive system variables, and the effect of changes that take place in these variables, on the system performance in order to determine the desired result.

3. Simulation experiments are done on the model, not on the system itself. Experimentation takes into consideration additional information during analysis that most quantitative models do not permit. In other words, simulation can be used to 'experiment' on a model of a real situation, without incurring the costs of operating on the system.

4. Simulation can be used as a pre-service test to try out new policies and decision rules for operating a system before running the risk of experimentation in the real system.

Disadvantages:

1. Simulation models are expensive and take a long time to develop. For example, a corporate planning model may take a long time to develop and may also prove to be expensive.

2. It is the trial-and-error approach that produces different solutions in repeated runs. This means it does not generate optimal solutions to problems.

3. The simulation model does not produce answers by itself. The user has to provide all the constraints for the solutions that he wants to examine.

4.7 Stochastic Simulation and Random Numbers

In simulation, probability distributions are used to quantify the outcomes in numerical terms by assigning a probability to each of the possible outcomes. For example, if you flip a coin, the set of possible outcomes is {H, T}. A random variable assigns a number to the possible occurrence of each outcome. In simulation, random variables are numerically controlled and are used to simulate elements of uncertainty that are defined in a model. This is done by generating (using the computer) outcomes with the same frequency as those encountered in the process being simulated. In this manner many experiments (also called simulation runs) can be performed, leading to a collection of outcomes that have a frequency (probability) distribution, similar to that of the model under study.

To use simulation, it is necessary to generate the sample random events that make up the model. This helps to use a computer to reproduce the process through which chance is generated in the actual situation. Thus, a problem that involves many interrelationships among random variables can be evaluated as a function of given parameters. Process generation (simulating chance processes) and modelling are therefore the two fundamental techniques that are needed in simulation. The most elementary and important type of process is the random process. This requires the selection of samples (or events) from a given distribution so that the repetition of this selection process would yield a frequency distribution of sample values that match the original distribution. These samples are generated through some mechanical or electronic device – called pseudo random numbers.

Alternately, it is possible to use a table of random numbers where the selection of number in any consistent manner would yield numbers that behave as if they were drawn from a uniform distribution. Random numbers can also be generated using random number generator (which are inbuilt feature of spread sheets and many computer languages) tables (see Appendix), a roulette wheel, etc. Random numbers between 00 and 99 are used to obtain values of random variables that have a known discrete probability distribution in which the random variable of interest can assume one of a finite number of different values. In some applications, however, the random variables are continuous, that is, they can assume any real value according to a continuous probability distribution.

4.8 Monte Carlo Simulation

The Monte Carlo simulation approach is used to incorporate the random behaviour of variable(s) of interest in a model. A formal definition of Monte Carlo Simulation is as follows:

- The Monte Carlo simulation technique involves conducting repetitive experiments on the model of the system under study, with some known probability distribution to draw random samples (observations) using random numbers.

The Monte Carlo simulation approach consists of following steps:

1. Setting up a probability distribution for variables to be analyzed.
2. Building a cumulative probability distribution for each random variable.
3. Generating random numbers and then assigning an appropriate set of random numbers to represent value or range (interval) of values for each random variable.
4. Conducting the simulation experiment using random sampling.
5. Repeating Step 4 until the required number of simulation runs has been generated.
6. Designing and implementing a course of action and maintaining control.

4.9 Random Number Generation

In Monte Carlo simulation we use a sequence of random numbers where,

(i) all numbers are equally likely

(ii) no patterns or recurrence relation is there in the sequence of numbers. These random numbers help in choosing random observations (samples) from the probability distribution.

Arithmetic computation: The nth random number $\mathbf{r_n}$, consisting of k-digits is generated by using multiplicative congruential method:

$\mathbf{r_n \equiv p.\ r_{n-1}}$ **(modulo m)**

where, **p** and **m** are positive integers, $\mathbf{p < m}$,

$\mathbf{r_{n-1}}$ is a k-digit number,

and **modulo m** means that $\mathbf{r_n}$ is the **remainder** when $\mathbf{p.\ rn_{-1}}$ is divided by **m**. This means, $\mathbf{r_n}$ and $\mathbf{p.\ rn_{-1}}$ differ by an integer multiple of **m**.

For generating random numbers, the first random number (or the **seed**) $\mathbf{r_0}$ is specified by the user. For example, to generate 2-digit random numbers we divide by 100 so that the remainder will be of 2- digits. This remainder is used as a random number.

Let p = 35, m = 100 and arbitrarily start with $r_0 = 57$. Since $m - 1 = 99$ is a 2-digit number,

therefore, it will generate 2-digit random numbers:

$r_1 = p\ r_0$ (modulo m) = 35 × 57 (modulo 100) = 1,995/100 = 95, remainder

$r_2 = p\ r_1$ (modulo m) = 35 × 95 (modulo 100) = 3,325/100 = 25, remainder

$r_3 = p\ r_2$ (modulo m) = 35 × 25 (modulo 100) = 875/100 = 75, remainder

Care must be taken while choosing r_0 and p for any given value of m, because a sequence of numbers generated, is determined by the input data for the simulation. The numbers generated in this process are **pseudo random numbers** because these are reproducible and hence, not random.

The congruential method can also be used to generate random numbers as decimal fraction between 0 and 1, with required number of digits. For this, the recurrence relation $\mathbf{u_n = r_n / m}$ is used to generate uniformly distributed decimal fraction between 0 and 1.

Computer generator: The random numbers that are generated by using computer software are uniformly distributed decimal fractions between 0 and 1. The software works on the concept of cumulative distribution function for random variables for which we seek to generate random numbers.

4.10 Inventory Problems

Example1: Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10 per cent defective products. Compare your answer with the expected probability.

Solution Given that 10 per cent of the total production is defective and 90 per cent is non-defective, if we have 100 random numbers (0 to 99), then 90 or 90 per cent of them represent non-defective products and the remaining 10 (or 10 per cent) of them represent defective products. Thus, the random numbers 00 to 89 are assigned to variables that represent non-defective products and 90 to 100 are assigned to variables that represent defective products.

If we choose a set of 2-digit random numbers in the range 00 to 99 to represent a packet of 6 products as shown below, then we would expect that 90 per cent of the time they would fall in the range 00 to 89.

Sample Number	Random Number
A	86 02 22 57 51 68
B	39 77 32 77 09 79
C	28 06 24 25 93 22
D	97 66 63 99 61 80
E	69 30 16 09 05 53
F	33 63 99 19 87 26
G	87 14 77 43 96 43
H	99 53 93 61 28 52
I	93 86 52 77 65 15
J	18 46 23 34 25 85

It may be noted that out of ten simulated samples 6 contain one or more defectives and 4 contain no defectives. Thus, the expected percentage of non-defective products is 40 per cent. However, theoretically the probability that a packet of 6 products containing no defective product is (0.9)6 = 0.53144 = 53.14%.

Example 2: A bakery keeps stock of a popular brand of cake. Previous experience shows the daily demand pattern for the item with associated probabilities, as given below:

Daily demand (number):	0	10	20	30	40	50
Probability:	0.01	0.20	0.15	0.50	0.12	0.02

Use the following sequence of random numbers to simulate the demand for next 10 days.

Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49.

Also estimate the daily average demand for the cakes on the basis of the simulated data.

Solution: Using the daily demand distribution, we first obtain a probability distribution as shown in Table 19.3.

Daily Demand	Probability	Cumulative Random Number	Probability Intervals
0	0.01	0.01	00
10	0.20	0.21	01–20
20	0.15	0.36	21–35
30	0.50	0.86	36–85
40	0.12	0.98	86–97
50	0.02	1.00	98–99

Next to conduct the simulation experiment for demand take a sample of 10 random numbers from a table of random numbers, which represent the sequence of 10 samples. Each random sample number represents a sample of demand.

The simulation calculations for a period of 10 days are given in Table 19.4.

Days	Random Number	Simulated Demand
1	40	30 ← because random number 40 falls in the interval 36–85
2	19	10 ← because random number 19 falls in the interval 01-20
3	87	40 and so on
4	83	30
5	73	30
6	84	30
7	29	20
8	09	10
9	02	10
10	20	10
		Total = 220

Expected demand = 220/10 = 22 units per day

Example - 3: A company manufactures around 200 mopeds. Depending upon the availability of raw materials and other conditions, the daily production has been varying from 196 mopeds to 204 mopeds, whose probability distribution is as given below:

Production/day:	196	197	198	199	200	201	202	203	204
Probability:	0.05	0.09	0.12	0.14	0.20	0.15	0.11	0.08	0.06

The finished mopeds are transported in a specially designed three-storied lorry that can accommodate only 200 mopeds. Using the following 15 random numbers: 82, 89, 78, 24, 53, 61, 18, 45, 23, 50, 77, 27, 54 and 10, simulate the mopeds waiting in the factory?

(a) What will be the average number of mopeds waiting in the factory?

(b) What will be the number of empty spaces in the lorry?

Solution:(a) Using production per day distribution, the daily production distribution is shown in Table.

Production/day	Probability	Cumulative Random Number	Probability Intervals
196	0.05	0.05	00 – 04
197	0.09	0.14	05 – 13
198	0.12	0.26	14 – 25

199	0.14	0.40	26 – 39
200	0.20	0.60	40 – 59
201	0.15	0.75	60 – 74
202	0.11	0.86	75 – 85
203	0.08	0.94	86 – 93
204	0.06	1.00	94 – 99

Based on the given 15 random numbers, simulation experiment of the production per day is show in Table

Days	Random Number	Production per Day	Number of Mopeds Waiting	Empty Space in the Lorry
1	82	202	2	—
2	89	203	3	—
3	78	202	2	—
4	24	198	—	2
5	53	200	0	—
6	61	201	1	—
7	18	198	—	2
8	45	200	—	—
9	04	196	—	4
10	23	198	—	2
11	50	200	—	—
12	77	202	2	—
13	27	199	1	—
14	54	200	—	—
15	10	197	—	3

Average number of mopeds waiting in the factory =

15 [2 + 3 + 2 + 1 + 2 + 1] = 1 moped (approximately)

Average number of empty spaces in the lorry = 13

15 = 0.86

4.11 Queuing Problems

Example 4: A dentist schedules all his patients for 30-minute appointments. Some of the patients take more 30 minutes some less, depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and time actually needed to complete the work:

Category of Time Required Probability Service		(minutes)
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
Check-up	15	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8.00 a.m. Use the following random numbers for handling the above problem: 40 82 11 34 25 66 17 79 .

Solution: The cumulative probability distribution and random number interval for service time are shown

Category of Service	Time Required (minutes)	Probability	Cumulative Probability	Random Number Interval
Filling	45	0.40	0.40	00–39
Crown	60	0.15	0.55	40–54
Cleaning	15	0.15	0.70	55–69
Extraction	45	0.10	0.80	70–79
Check-up	15	0.20	1.00	80–99

The parameters of a queuing system such as arrival pattern of customers, service time, waiting time, for the given problem, are shown in Tables

Patient Number	Scheduled Arrival	Random Number	Category of Service	Service Time (minutes)
1	8.00	40	Crown	60
2	8.30	82	Check-up	15
3	9.00	11	Filling	45
4	9.30	34	Filling	45
5	10.00	25	Filling	45
6	10.30	66	Cleaning	15
7	11.00	17	Filling	45
8	11.30	79	Extraction	45

Time (Patient Number)	Event	Patient Number (Time to Exit)	Waiting (Patient Number)
8.00	1 arrive	1 (60)	–
8.30	2 arrive	1 (30)	2
9.00	1 departs; 3 arrive	2 (15)	3
9.15	2 depart	3 (45)	–
9.30	4 arrive	3 (30)	4
10.00	3 depart; 5 arrive	4 (45)	5
10.30	6 arrive	4 (15)	5, 6
10.45	4 depart	5 (45)	6
11.00	7 arrive	5 (30)	6, 7
11.30	5 depart; 8 arrive	6 (15)	7, 8
11.45	6 depart	7 (45)	8
12.00	End	7 (30)	8

The dentist was not idle even once during the entire simulated period. The waiting times for the patients were as follows:

Patient	Arrival Time	Service Starts at	Waiting Time (minutes)
1	8.00	8.00	0
2	8.30	9.00	30
3	9.00	9.15	15
4	9.30	10.00	30
5	10.00	10.45	45
6	10.30	11.30	60
7	11.00	11.45	45
8	11.30	12.30	60
			280

The average waiting time = 280/8 = 35 minutes.

Example 5: XYZ company is considering the issue of marketing a new product. The fixed cost required in the project is Rs 4,000. Three factors are uncertain, viz., the selling price, variable cost and the annual sales volume. The product has a life of only one year. The management has the data on these three factors as under:

Selling Price (Rs)	Probability	Variable Cost (Rs)	Probability	Sales Volume (Units)	Probability
3	0.2	1	0.3	2,000	0.3
4	0.5	2	0.6	3,000	0.3
5	0.3	3	0.1	5,000	0.4

Considering the following sequence of thirty random numbers: 81, 32, 60, 04, 46, 31, 67, 25, 24, 10, 40, 02, 39, 68, 08, 59, 66, 90, 12, 64, 79, 31, 86, 68, 82, 89, 25, 11, 98, 16.

Using the sequence (First 3 random numbers for the first trial, etc.) simulate the average profit for the above project on the basis of 10 trails.

Solution The cumulative probability distribution and random number interval for selling price, variable cost and sales volume are shown below:

Selling Price (Rs)	Probability	Cumulative Probabilities	Random Numbers Interval
3	0.2	0.2	00—19
4	0.5	0.7	20—69
5	0.3	1.0	70—99
Variable cost (Rs)			
1	0.3	0.3	00—29
2	0.6	0.9	30—89
3	0.1	1.0	90—99
Sales volumes (Units)			
2,000	0.3	0.3	00—29
3,000	0.3	0.6	30—59
5,000	0.4	1.0	60—99

The simulation experiment sheet for finding average profit is shown in Table

Number of Trials	Random Number	Selling Price (Rs)	Random Number	Variable Cost (Rs)	Random Number	Sales Volume ('000 units)
1	81	5	32	2	60	5
2	04	3	46	2	31	3
3	67	4	25	1	24	2

4	10	3	40	2	02	2
5	39	4	68	2	08	2
6	59	4	66	2	90	5
7	12	3	64	2	79	5
8	31	4	86	2	68	5
9	82	5	89	2	25	2
10	11	3	98	3	16	2

Trial Number	**Profit = (Selling Price – Variable Cost) × Sales Volume – Fixed Cost**	
1	(5 – 2) × 5,000 – 4,000	= 11,000
2	(3 – 2) × 3,000 – 4,000	= (– 1000)
3	(4 – 1) × 2,000 – 4,000	= 2,000
4	(3 – 2) × 2,000 – 4,000	= (– 2,000)
5	(4 – 2) × 2,000 – 4,000	= 0
6	(4 – 2) × 5,000 – 4,000	= 6,000
7	(3 – 2) × 5,000 – 4,000	= 1,000
8	(4 – 2) × 5,000 – 4,000	= 6,000
9	(5 – 2) × 2,000 – 4,000	= 2,000
10	(3 – 3) × 2,000 – 4,000	= (– 4,000)
		21,000
	Average profit per trial	= 21,000/10 = Rs 21,00

4.12 INVESTMENT PROBLEMS

Example 6: An investment agency wants to analyze the investment projects based on three factors: market demand in units' price per unit minus cost per unit, and investment required. These factors are believed to be independent of each other. In analyzing a new consumer product, the Corporation estimates the following probability distributions:

Annual Demand		**Price minus Cost per Unit**		**Investment Required**	
Units	**Probability**	**Rs**	**Probability**	**Rs**	**Probability**
20,000	0.05	3.00	0.10	17,50,000	0.25
25,000	0.10	5.00	0.20	20,00,000	0.50
30,000	0.20	7.00	0.40	25,00,000	0.25
35,000	0.30	9.00	0.20		

40,000	0.20	10.00	0.10
45,000	0.10		
50,000	0.05		

Using the simulation process, repeat the trial 10 times, compute the return on investment for each trial taking these three factors into account. What is the most likely return?

Solution: The return per annum can be computed by the following expression

Return (R) = [(Price – Cost) × Number of units demanded] /Investment

Developing a cumulative probability distribution, corresponding to each of the three factors, an appropriate set of random numbers is assigned to represent each of the three factors, as shown in the following Tables:

Annual Demand	Probability	Cumulative Probability	Random Number
20,000	0.05	0.05	00–04
25,000	0.10	0.15	05–14
30,000	0.20	0.35	15–34
35,000	0.30	0.65	35–64
40,000	0.20	0.85	65–84
45,000	0.10	0.95	85–94
50,000	0.05	1.00	95–99

Price minus	Probability Cost per Unit	Cumulative	Random Number Probability
3.00	0.10	0.10	00–09
5.00	0.20	0.30	10–19
7.00	0.40	0.70	20–69
9.00	0.20	0.90	70–89
10.00	0.10	1.00	90–99

Investment Required	Probability	Cumulative	Random Number Probability
17,50,000	0.25	0.25	00–24
20,00,000	0.50	0.75	25–74
25,00,000	0.25	1.00	75–99

The simulation worksheet is prepared for 10 trials. The simulated return (R) is also calculated by using the formula for R, as stated before. The simulation worksheet is shown in the following Table:

Trials	Random Number for Demand 100	Simulated Demand ('000)	Random Number for Profit (Price – Cost) per Unit	Simulated Profit	Random Number for Investment	Simulated Investment ('000)	Simulated return (%): Demand × Profit per unit × Investment
1	28	30	19	5.00	18	1,750	8.57
2	57	35	07	3.00	61	2,000	5.25
3	60	35	90	10.00	16	1,750	20.00
4	17	30	02	3.00	71	2,000	4.50
5	64	35	57	7.00	43	2,000	12.25
6	20	30	28	5.00	68	2,000	7.50
7	27	30	29	5.00	47	2,000	7.50
8	58	35	83	9.00	24	1,750	18.00
9	61	35	58	7.00	19	1,750	14.00
10	30	30	41	7.00	97	2,500	8.40

As shown in Table, the highest likely return is 20 per cent, which corresponds to the annual demand of 35,000 units yielding a profit of Rs 10 per unit and investment required is Rs. 17, 50,000.

□□□

ଓଡ଼ିଶା ରାଜ୍ୟ ମୁକ୍ତ ବିଶ୍ୱବିଦ୍ୟାଳୟ, ସମ୍ବଲପୁର
Odisha State Open University, Sambalpur
www.osou.ac.in
e-mail: info@osou.ac.in