BBA 11
Block - 2

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Odisha State Open University
Sambalpur

BBA

BACHELOR OF
BUSINESS ADMINISTRATION

Quantitative Techniques for Management

TRANSPORTATION AND ASSIGNMENT MODELS

ଓଡ଼ିଶା ରାଜ୍ୟ ମୁକ୍ତ ବିଶ୍ୱବିଦ୍ୟାଳୟ, ସମ୍ବଲପୁର, ଓଡ଼ିଶା
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Quantitative Techniques for Management
BBA-11

Block

2 TRANSPORTATION AND ASSIGNMENT MODELS

Unit-1
Elementary Transportation Problem

Unit-2
Elementary Assignment Problem

BBA – Course - 11
Block - 2

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Unit-1
Elementary Transportation Problem

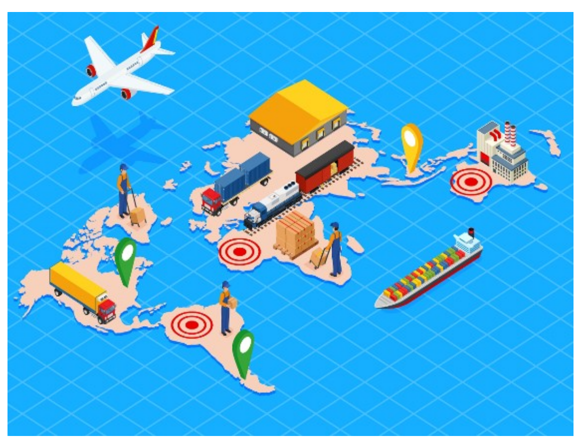
Structure :

1.1 Learning Objectives

Thisunit deals with optimization technique in transportation problem.A transportation problem can be solved by several techniques such asN.W. Corners Rule, Least-cost method, Vogel's Approximation method and Modified Distribution method. After completion of this unit, the reader is expected to be able to formulate a transportation problem and solve the same.

1.2 Introduction

This is a special kind of Linear Programming Problem (LPP) where the objective is to transport the goods or commodities from the places of origin to the various destinations at a minimal cost.To achieve this,we must know the amount available at each place of origin and the demand at each destination.In addition to this we must also know the transportation cost of one unit of goods or commodities from several places of origins to several destinations i.e., transportation cost per unit.



1.3 Formulation of a Transportation Problem

Consider a transportation problem that has 'm' origins where each origin has a_i (i=1,2, 3, …., m) units of homogeneous goods (or commodities) and 'n' destinations each requiringb_j(j=1, 2, 3, …, n) units of goods. Let x_{ij} be the quantity of goods transported from origini to destination j. LetC_{ij} be the cost of transporting a unit value from the place of origin i to the destination j. The said transportation problem can be shown in a tabular form known as **transportation table** as follows:

Origin	Destination							
		1	2	…	j	…	n	Supply
	1	x_{11} (C_{11})	x_{12} (C_{12})	…	x_{1j} (C_{1j})	…	x_{1n} (C_{1n})	a_1
	2	x_{21} (C_{21})	x_{22} (C_{22})	…	x_{2j} (C_{2j})	…	x_{2n} (C_{2n})	a_2
	…	…	…	…	…	…	…	…
	i	x_{i1} (C_{i1})	x_{i2} (C_{i2})	…	x_{ij} (C_{ij})	…	x_{in} (C_{in})	a_i
	…	…	…	…	…	…	…	…
	m	x_{m1} (C_{m1})	x_{m2} (C_{m2})	…	x_{mj} (C_{mj})	…	x_{mn} (C_{mn})	a_m
	Demand	b_1	b_2	…	b_j	…	b_n	$\sum_{j=1}^{n} b_j = \sum_{i=1}^{m} a_i$

The above transportation problem can be represented in the form of a linear programing problem as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to the constraints,

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ (Column total) and}$$
$$\sum_{j=1}^{n} x_{ij} = a_i \text{ (Row total)}$$

Non negative conditions,

$$x_{ij} \geq 0 \text{ for all i and j.}$$

NB:
The above transportation problem is said to be balanced if the total amount of supply is same as total amount of demand i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

1.4 Terminology

Before solving a transportation problem let us discuss some basic terminology.

14.1 Feasible Solution (FS)

A set of non-negative allocations ($x_{ij} \geq 0$) which satisfies the row and column sums is said to be a feasible solution.

1.4.2 Basic feasible solution (BFS)

In a transportation problem with m rows and n columns, if the number of non-negative allocations is equal to m+n-1, then the feasible solution is known as a basic feasible solution.

1.4.3 Optimal solution (OS)

In a transportation problem, if a feasible solution optimizes the transportation cost, then it is known as an optimal solution.

1.4.4 Non-degenerate basic feasible solution (NDBFS)

In a transportation problem with m rows and n columns, the feasible solution is said to be non-degenerate if it contains m+n-1 occupied cells.

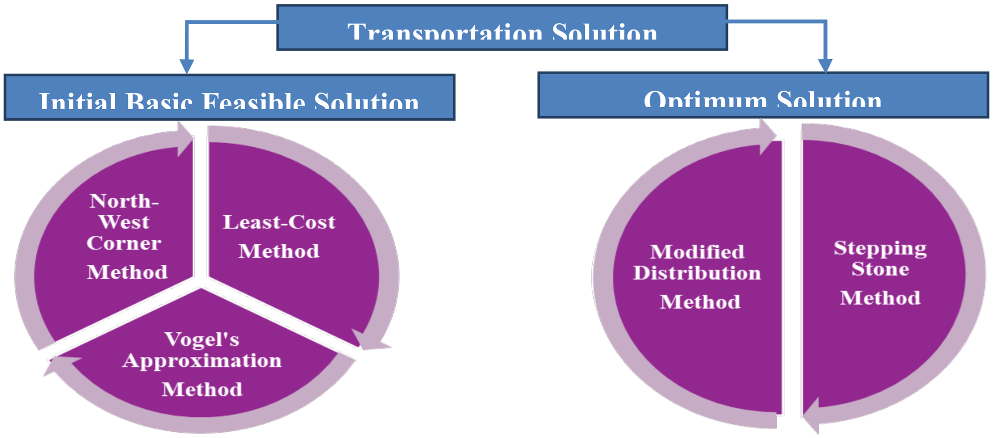
1.5 Algorithm to Solve a Transportation Problem

A transportation problem can be solved using the following algorithm:

- Step-1: Covert the problem into the form of a Linear Programming Problem(LPP).
- Step-2: Setup the LPP into a transportation table.
- Step-3: Examine for equality of total supply and demand. If unequal, then introduce a dummy with zero cost and supply or demand as the +ve difference between them.
- Step-4: Find the initial BFS which satisfies all supply and demand conditions.
- Step-5: Check for an improved transportation plan with lower cost.
- Step-6: If the solution is not optimum , modify the plan by including unoocupied cells for an improved solution.
- Step-7: Repeate step-4 until and unless no further improvement is posibble.

1.6 Types of Solution of a Transportation Problem

The solutions of a transportation problem are of two types:Initial Basic Feasible solution and Optimal solution. Thevarious methods that can be used to find an initial basic feasible solution and optimal solutionare shown in the figure below:



1.7 North-West Corner Method (NWCS)

This is the most general and effective method of finding an initial basic feasible solution. Different steps of this method are as follows:

Step-1
- Identify the Upper left (North-West) corner cell (1, 1) of the transportation table.

Step-2
- Allocate the cell (1, 1) in such a way that the capacity of the first row is exhausted by supply or the demand in first coloumn is satisfied i.e., min(a_1, b_1).

Step-3
- If the demand is greater than the supply in cell one (i.e., $b_1 > a_1$) then move vertically downwards to the second row and allocate with magnitude min(a_2, $b_1 - x_{11}$) in the cell (2,1).. Then go to step 6.

Step-4
- If the demand is lesser than the supply in cell one (i.e., $b_1 < a_1$) then move horizontally right to the second coloumn and allocate with magnitude min($a_1 - x_{11}$, b_2) in the cell (1, 2). Then go to step 6.

Step-5
- If the demand and supply are same in cell one (i.e., $b_1 = a_1$) then Second allocation can be allocated with the following magnitude:

  min($a_1 - a_1$, b_1) = 0 in the cell (1, 2) or
  min(a_2, $b_1 - b_1$) = 0 in the cell (2, 1)

  Then go to step 6.

Step-6
- Repeat all the steps from step-1 with the aim to reach lower right corner of the transportation table until all demand and supply are satisfied.

Example:1 (NWCM)

The cost table and rim requirement table of a transportation problem is given below. Find out its initial basic feasible solution by using north west corner method.

Origin ↓ / Destination →	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

Solution :

As $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 34$, a feasible solution is existsfor the given transportation problem. Noe we can obtain its initial feasible solution as follows:

Starting from north west corner,

- The 1st allocation is made in the cell (1, 1) whose magnitude is x_{11} = min (5,7) = 5.
- The 2nd allocation is made in the cell (2, 1) with magnitude x_{21} = min (8, 7-5) = 2.
- The 3rd allocation is made in the cell (2, 2) with magnitude x_{22} = min (8-2, 9) = 6.
- The 4th allocation is made in the cell (3, 2) with magnitude x_{32} = min (7, 9-6) = 3.
- The 5th allocation is made in the cell (3, 3) with magnitude x_{33} = min (7-3, 14) = 4.
- The final allocation is made in the cell (4, 3) with magnitude x_{43} = min (14, 18-4) = 14.

Origin ↓ / Destination →	D_1	D_2	D_3	Supply
O_1	2 (5)	7	4	~~5~~ 0
O_2	3 (2)	3 (6)	1	~~8~~ ~~6~~ 0
O_3	5	4 (3)	7 (4)	~~7~~ ~~4~~ 0
O_4	1	6	2 (14)	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	34

Now, the initial basic feasible solution to the given transportation problem is given by:

$x_{11} = 5$, $x_{21} = 2$, $x_{22} = 6$,$x_{32} = 3$,$x_{33} = 4$, $x_{43} = 14$.

Hence the total minimum cost = (5×2) + (2×3) + (6×3) + (3×4) + (4×7) + (14×2)

= 10 + 6 + 18 + 12 + 28 + 28 = 102 rupees.

1.8 Least Cost or Matrix Minima Method (LCM or MMM)

As the name suggests this method takes the smallest unit cost in the cost matrix of the transportation table . The steps are summarized as follows :

Step-1
- Point out the smallest cost C_ij (say) in the cost matrix of the transportation table . If it is not unique make a random choice between available minimum values.

Step-2
- Then allocate min(a_i, b_j) in the cell (i, j) of the transportation table.

Step-3
- If min(a_i, b_j) = a_i then cross off the i^th row of the transportation table and reduce the value b_j by a_i . Then go to step 6.

Step-4
- If min(a_i, b_j) = b_j then cross off the j^th coloumn of the transportation table and reduce the value a_i by b_j . Then go to step 6.

Step-5
- If min(a_i, b_j) = b_j = a_i then cross off either i^th row or j^th coloumn of the transportation table. Then go to step 6.

Step-6
- Repeat all the steps from step-1 with the aim of satisfying all requirements.

Example : 2 (LCM or MMM)

Let there be a transportation problem with following cost table. Obtain an initial basic feasible solution using Least cost method or Matrix minima method.

Origin ↓ / Destination →	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution :

As $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 24$, i.e as total supply is equal to the total demand, the given transportation problem has a feasible solution.

We can obtain its initial feasible solution by the least cost method or matrix minima method as follows:

Applying the least cost method;

- The 1st demand is satisfied at the destination D_1 and the allocation is made in the cell (3,1) with magnitude x_{31} = min (4, 10) = 4. Now, we can delete the 1st column from the table as the demand is exhausted.
- The 2nd demand is satisfied at the destination D_4 and the allocation is made in the cell (2,4) with magnitude x_{24} = min (6, 8) =6. Now, we can delete the 4th column from the table as the demand is exhausted.
- Then from the reduced cost table, the 3rd allocation is made in the cell (3, 3) with the magnitude x_{33} = min (8, 6) = 6.
- The 4th allocation is made in the cell (2,3) with magnitude x_{23} = min (2, 2) = 2. Then, we can delete the 2nd row and 3rd column from the table as the demand is exhausted.
- The final allocation is made in the cell (1, 2) with magnitude x_{12} = min (6, 6) = 6 and we can delete the 1st row and 2nd column from the table as the demand is exhausted.

Now the initial basic feasible solution is obtained as all the rim requirement has been satisfied.

Origin ↓ / Destination →	D_1	D_2	D_3	D_4	Supply
O_1	1	2 (6)	3	4	~~6~~ 0
O_2	4	3	2 (2)	0 (6)	~~8~~ ~~2~~ 0
O_3	0 (4)	2	2 (6)	1	~~10~~ ~~6~~ 0
Demand	~~4.~~ 0	~~6~~ 0	~~8~~ ~~2~~ 0	~~6~~ 0	34

Now, the initial basic feasible solution to the given transportation problem is given by; x_{12} = 6, x_{23} = 2, x_{24} = 6, x_{31} = 4, x_{33} = 6. As the total number of occupied cells is 5, which is less than m+n+1, so we have a degenerate solution.

Hence the minimum transportation cost obtained as:

$$Cost_{min} = (6\times2) + (2\times2) + (6\times0) + (4\times0) + (6\times2)$$

$$= 12 + 4 + 12 = 28 \text{ rupees.}$$

1.9 Vogel's Approximation Method

Vogel's approximation is superior over the previous two methods for finding an initial basic feasible solution as it not only takes the least cost but also takes its nearest exceeded cost into account. The steps involve in this method are as follows:

Step-1
- Point out the smallest cost C_{ij} (say) and nearest smallest cost $C_{ij}^{'}$(say) for every row and column of the transportation table . Find out the difference between them for both rows and columns known as penalty cost.

Step-2
- Then choose the maximum penalty cost. If it is more than one or there is a tie case then we can choose any one of them arbitarily from any row or column.

Step-3
- After getting the maximum penalty cost find out the cell having least cost in those rows or columns.

Step-4
- Allocate these cells as much as possible depending on the capacity and requirement.

Step-5
- Delete the rows and coloumns those are fully exhausted. Again compute the row and coloumn penealty to make a reduced transportation table.

Step-6
- Repeat all the steps from step-1 with aiming that all the requirements are satisfied.

The initial basic feasible solution from VAM method is very close to the optimum solution

Example:3 (VAM)

Obtained the initial basic feasible solution for the following transportation problem by using VAM or Vogel's approximation method.

Origin ↓ / Destination →	D_1	D_2	D_3	D_4	Supply
O_1	11	13	17	14	250
O_2	16	18	14	10	300
O_3	21	24	13	10	400
Demand	200	225	275	250	950

Solution:

As $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 950$, the given transportation problem is balanced and it has a feasible solution.

The first step is to find the row and column penalty (P_I)which is the difference between least two costs. There are two maximum penalties (i.e., 5) from 1st column (D_1) and 2nd Column (D_2). Let us choose the first column (D_1) arbitrarily. From this we column choose the least cost cell (1, 1) and allocate this with the magnitude min (250, 200) = 200. As the demand is exhausted, we delete the 1st column.

Allocation - I

Destination→ Origin↓	D_1	D_2	D_3	D_4	Supply	P_I
O_1	11 (200)	13	17	14	~~250~~ 50	2
O_2	16	18	14	10	300	4
O_3	21	24	13	10	400	3
Demand	~~200~~ 0	225	275	250		
P_I	5↑	5	3	0		

As, the 1st column is deleted, the row penalty P_{II} is change and column penaltyP_{II}remains the same.

Allocation - II

Destination→ Origin↓	D_2	D_3	D_4	Supply	P_{II}
O_1	13 (50)	17	14	~~50~~ 0	1
O_2	18	14	10	300	4
O_3	24	13	10	400	3
Demand	~~225~~ 175	275	250		
P_{II}	5↑	1	0		

Continuing in the same manner we get the next allocations as given in the table below:

Allocation-III

Destination→ Origin↓	D_2	D_3	D_4	Supply	P_{III}
O_2	18 (175)	14	10	~~300~~ 125	4
O_3	24	13	10	400	3
Demand	~~175~~ 0	275	250		
P_{III}	6↑	1	0		

Allocation-IV

Destination→ Origin↓	D_3	D_4	Supply	P_{IV}
O_2	14	10 (125)	~~125~~ 0	← 4
O_3	13	10	400	3
Demand	275	~~250~~ 125		
P_{IV}	1	0		

Allocation-V

Destination → Origin↓	D_3	D_4	Supply	P_V
O_3	13 (275)	10	~~400~~ 125	3
Demand	~~275~~ 0	125		
P_V	13↑	10		

Allocation-VI

Destination → Origin↓	D_4	Supply	P_{VI}
O_3	10 (125)	~~125~~ 0	←10
Demand	~~125~~ 0		
P_{VI}	10		

Finally, we have reached at the initial basic feasible solution shown in the table below:

Destination→ Origin↓	D_1	D_2	D_3	D_4	Supply
O_1	11 (200)	13 (50)	17	14	250
O_2	16	18 (175)	14	10 (125)	300
O_3	21	24	13 (275)	10 (125)	400
Demand	200	225	275	250	

The solution is a non-degenerate basic feasible solution as there are 6 positive independent allocations which are equal to m+n-1 = 3+4-1 = 6.

Hence, the final transportation cost is obtained as:

$Cost_{min}$ = 200×11 + 50×13 + 175×18 + 125×10 + 275×13 + 125×10 = 12075 rupees.

1.10 Modified Distribution or MODI Method

After getting initial basic feasible solution, next task is to test the optimality of the solution. The Modified distribution method or MODI method is generally used to test the optimality of the solution. Thesteps involved in testing the optimality are described as follows:

Step-1
- Find out the initial basic feasible solution of a given transportation problem by any one method amog the earlier discussed methods.

Step-2
- Figure out the pairs of numbers u_i and v_j for each row and column that satisfies $u_i + v_j = c_{ij}$. The row or coloumn with maximum number of allocations assigned to them as 0. In more than one maximum cases we can choose any one arbitarily.

Step-3
- For every empty or unoccupied cell find the sum of u_i and v_j .

Step-4
- Find the net evaluation value $\Delta\_ij= c\_ij-(u\_i+v\_j)$ for each empty or unoccupied cell.

Step-5
- Then the optimality conclusion can be drawn as follows: i) If $\Delta\_ij> 0$ then the solution is said to be optimum with unique solution. ii) If $\Delta\_ij\geq 0$ then the solution is said to be optimum but not an unique solution. iii) If $\Delta\_ij< 0$ then the solution is not optimum (Then go to step-6).

Step-6
- Select an empty cell having most negative value of Δ_ij. From this cell draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign + and - sign alternatively. Then find the minimum allocation from the cell having negative sign. this allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

Step-7
- Again repeat step-2 to 5 till an optimum solution is attatined.

Example:4 (MODI)

Obtain the initial basic solution and test the optimality of the solution of the following transportation problem using modified distribution or MODI method.

Destination→ Source↓	A	B	C	D	Supply
P	21	16	25	13	11
Q	17	18	14	23	13
R	32	17	18	41	19
Demand	6	10	12	15	43

Solution :

First, we find the initial basic solution by the help of any one method discussed earlier. Let's use by VAM method. Since $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 43$, this transportation problem is balanced and a feasible solution also exists. Now, the table for row and column penalty can be constructed as follows:

Destination→ Source↓	A	B	C	D	Supply	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
P	21	16	25	13 (11)	~~11~~ 0	3	--	--	--	--	--
Q	17 (6)	18	14 (3)	23 (4)	~~13 9 3~~ 0	3	3	3	3	--	--
R	32	17 (10)	18 (9)	41	~~19 10~~ 0	1	1	1	1	1	17
Demand	~~6~~ 0	~~10~~ 0	~~12~~ ~~9~~ 0	~~15~~ ~~4.~~ 0	43						
P_I	4	1	4	10↑							
P_{II}	15	1	4	18↑							
P_{III}	15↑	1	4	--							
P_{IV}	--	1	4↑	--							
P_V	--	17	18↑	--							
P_{VI}	--	17↑	--	--							

In the above table the number of non-negative independent allocations is 6, which is equal to m+n-1 = 3+4-1 = 6. So, here we will get a non-degenerate basic feasible solution.

The initial transportation cost becomes ;

Cost = 11×13 + 3×14 + 4×23 + 6×17 + 10×17 + 9×18 = 711 rupees.

Now, we can apply MODI method to determine the optimum solution. We determine a set of numbers u_i and v_j for each row and column such that $u_i + v_j = c_{ij}$ for each occupied cell. By starting we give $u_2 = 0$ as the 2nd row has maximum number of allocations.

$$c_{21} = u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17$$

$$c_{23} = u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14$$

$$c_{24} = u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23$$

$$c_{14} = u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = -10$$

$$c_{33} = u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4$$

$$c_{32} = u_3 + v_2 = 17 = 4 + v_2 = 17 \Rightarrow v_2 = 13$$

Then, we find the sum of u_i and v_j for all empty cells and keep it at the bottom left corner of the respective cell. In the next step, we obtain the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for every unoccupied cell and keep it at the bottom right corner of the respective cell.

Destination→ Source↓	A	B	C	D	u_i
P	21 7 14	16 3 13	25 4 21	13 (11)	u_1= -10
Q	17 (6)	18 13 5	14 (3)	23 (4)	u_2= 0
R	32 21 09	17 (10)	18 (9)	41 25 16	u_3 = 4
v_j	v_1=17	v_2=13	v_3=14	v_4=23	

Since all the $\Delta_{ij} > 0$, the solution of this transportation problem is optimal and unique. Now, the optimum solution becomes;

$x_{14} = 11$, $x_{21} = 6$, $x_{23} = 3$, $x_{24} = 4$, $x_{32} = 10$ and $x_{33} = 9$.

Thus, we obtain the minimum transportation cost as;

$Cost_{min} = 11\times13 + 6\times17 + 3\times14 + 4\times23 + 10\times17 + 9\times18 = 711$ rupees

1.11 Some Special Cases

A transportation problem is said to be balanced if the sum of all the demands is equal to the sum of all the supplies. However, in real situations the above condition may not be satisfied always i.e., the total demand may not be equal to the total supply. The objective may also be maximization. There may be some restrictions on the transportation roots. Such kind of problems may be categorized as follows:

a. Unbalanced transportation case
b. Multiple / Alternative optimal solutions
c. Maximization transportation case
d. Prohibited transportation roots

Now, let us discuss these above cases one by one:

a. Unbalanced transportation case

The necessary and sufficient condition for existence of a feasible solution to a general transportation problem (with m rows and n columns) is that the total demandis equal to total supply i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Where; a_i be the availability at i^{th} origin and $b_j$be the requirement at j^{th} destination. This problem is also known as a balanced transportation problem.

But, sometimes the total distance may not be equal to the total supply.In such cases, the problem is said to be unbalanced.There are two situations which may occur in an unbalanced transportation problem and they are;

$$i)\ \sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \ and\ ii)\ \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$$

Under case (i) we have to introduce a dummy destination in the cost matrix. Then set the transportation cost to this dummy destinationas equals to zero. So, the requirement at this dummy destination is equal to:

$$\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$

Under case (ii) we have to introduce a dummy source in the cost matrix. Then set the transportation cost from this dummy source to any destination equals to zero. So, the availability at this dummy source is equal to:

$$\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$

b. Multiple / alternative solutions

The existence of alternative optimal solutions can be determined by an inspection of the opportunity cost, $Z_{ij} - C_{ij}$ for the unoccupied cells. If an unoccupied cell in an optimal solution has opportunity cost zero, then an alternative optimal solution can be formed with another set of allocations without increasing total transportation cost.

c. Maximization case

In general, transportation method is used for a minimization problem. However, it may also be used to solve problems in which the objective is to maximize, when we consider the unit profit (or payoff) P_{ij} instead of the unit cost C_{ij}associated with each route (i, j). The procedure for solving such problems is summarized below:

Step-1: Convert the given problem into that of minimization by replacing each element of the transportation table by its difference from the maximum element of the table.

Step-2: Find an initial feasible solution using any of the three methods.

Step-3: Use MODI method for finding an optimal solution.

d. Prohibited transportation roots

Sometimes situations may arise when it is not possible to transport goods from certain sources to certain destinations.For example: hazards (snow, flood, etc.), road traffic regulations,etc. Such type of problems can be handled by assigning a very large cost say M to that route.

Example: 5 (Unbalanced problem)

Obtain theinitial basic feasible solution for the following transportation problem, where the shipping cost from plant to market is given and the quantity at each market and available at each plant also given.

Solution :

Plant↓ / Market→	W1	W2	W3	W4	Availability
F1	11	20	7	8	50
F2	21	16	10	12	40
F3	8	12	18	9	70
Requirements	30	25	35	40	

Here total requirement of the markets = 30+25+35+40=130 and total availability at the plants= 50+40+70=160. Since the total availability at the plants is 30 more than the total requirement in four markets W1, W2, W3 and W4 the transportation problem is unbalanced. Let us convert this problem into a balance one by introducing a dummy market W5 with requirement 30 such that the cost of transportation from plant to this market W5 is zero.

Thus, the balanced transportation problem is given by the following table:

Plant↓ / Market→	W1	W2	W3	W4	W5	Availability
F1	11	20	7	8	0	50
F2	21	16	10	12	0	40
F3	8	12	18	9	0	70
Requirements	30	25	35	40	30	160 (Total)

By using V.A.M. we get the following basic feasible solution of the problem. Now, the table for row and column penalty can be constructed as follows:

Destination → Source↓	W1	W2	W3	W4	W5	Supply	P_I	P_{II}	P_{III}	P_{IV}	P_V	P_{VI}
F1	11	20	7 (35)	8 (15)	0	~~5015~~0	7	1	1	1	1	8
F2	21	16	10	12 (10)	0 (30)	~~4010~~ 0	10 ←	2	2	2	2	12 ←

F3	8 (30)	12 (25)	18	9 (15)	0	~~704515~~ 0	8	1	1	9 ←	-	-
Demand	~~30~~ 0	~~25~~ 0	~~35~~ 0	~~40~~ ~~25~~ ~~15~~ 0	~~30~~ 0	160						
P_I	3	4	3	1	0							
P_{II}	3	4↑	3	1	-							
P_{III}	3↑	-	3	1	-							
P_{IV}	-	-	3	1	-							
P_V	-	-	3↑	2	-							
P_{VI}	-	-	-	2	-							

In the above table the number of independent non-negative allocations is 7 which is equal to m+n-1 = 3+5-1 = 7. So, here we will get a non-degenerate basic feasible solution.

The initial basic feasible solution to the giventransportation problem cost becomes;

$$\text{Cost} = 8\times30 + 12\times25 + 7\times35 + 8\times15 + 12\times10 + 9\times15 + 0\times30$$

$$= 240 + 300 + 245 + 120 + 120 + 135 = 1160 \text{ rupees.}$$

Example: 6 (Maximization problem)

There are three manufacturing companies supply its product to four dealers. The following profit table involves,per unit profit associated with different companies and different dealers.

Dealer→ Company↓	D1	D2	D3	D4	Capacity
C1	6	6	6	4	1000
C2	4	2	4	5	700
C3	5	6	7	8	900
Requirement	900	800	500	400	2600

Obtain the initial basic feasible solution to maximize the total profit.

Solution :

First, we have to convert the given profit table into its equivalencelosstable. The loss table can be obtained by subtracting all its elements from the highest element i.e., 8. Now, we can have the following loss table;

Dealer→ Company↓	D1	D2	D3	D4	Capacity
C1	2	2	2	4	1000
C2	4	6	4	3	700
C3	3	2	1	0	900
Requirement	900	800	500	400	2600

Here we use VAM to get the initial basic feasible solution. Now, the table for row and column penalty can be constructed as follows:

Destination→ Source↓	D1	D2	D3	D4	Supply	P_I	P_{II}	P_{III}
C1	2 (900)	2 (100)	2	4	~~1000~~ ~~100~~ 0	0	0	0
C2	4	6 (200)	4 (500)	3	~~700~~ ~~200~~ 0	1	0	2 ←
C3	3	2 (500)	1	0 (400)	~~900~~ ~~500~~ 0	1	1	1
Demand	~~900~~ 0	~~800~~ ~~600~~ ~~500~~ 0	~~500~~ 0	~~400~~ 0	2600			
P_I	1	0	1	3↑				
P_{II}	1↑	0	1	-				
P_{III}	-	0	1	-				

In the above table the number of independent non-negative allocated cells is 6 which is equal to m+n-1 = 3+4-1 = 6. So, here we can get a non-degenerate basic feasible solution.

The initial basic feasible solution to the given transportation problem becomes;

$$\text{Cost} = 2\times900 + 2\times200 + 6\times200 + 4\times500 + 2\times500 + 0\times400$$

$$= 1800+400+1200+2000+1000= 64000 \text{ rupees.}$$

Example : 7 (Prohibited transportation roots)

Consider the following transportation problem where the cost of transportation in third source to the third destination is missing. Find out the number of units should be transported from different sourced to different destinations so that the total transportation cost from all the sources to all the destinations is minimum.

Destinations → Sources↓	D1	D2	D3	Capacity
S1	2	2	3	10
S2	4	1	2	15
S3	1	2	--	40
Demand	20	15	30	65

Answer :

As the transportation cost in the cell (S3, D3)is missing we assign a very large cost (say M) to it. Then by using VAM we can obtain an initial basic feasible solution. The penalty table for row and column can be constructed as follows:

Destination→ Source↓	D1	D2	D3	Supply	P_I	P_{II}	P_{III}
S1	2	2	3 (10)	~~10~~ 0	0	1	3
S2	4	1	2 (15)	~~15~~ 0	1	1	2
S3	1 (20)	2 (15)	M (5)	~~40~~ ~~20~~ ~~5~~ 0	1	M ←	M ←
Demand	~~20~~ 0	~~15~~ 0	30	65			
P_I	1↑	1	1				
P_{II}	-	1	1				
P_{III}	-	-	1				

In the above table the number of independent non-negative allocated cells is 5 which is equal to m+n-1 = 3+3-1 = 5. So, here we can get a non-degenerate basic feasible solution.

The initial basic feasible solution to the given transportation problem becomes;

Cost =3×10 + 2×15 + 1×20 +2×15 + M×5 (Pseudo)= 95 rupees.

N.B.: We can neglect the pseudo basic feasible solution.

Example : 8 (Multiple / Alternative solutions)

Arefrigerator manufacturingcompany has two different plants located at Bhubaneswar and Cuttack with a monthly capacity of 200 and 100 units respectively.The company supplies its refrigerator to its four godown situated at Sambalpur, Jagatsinghpur, Jajpur and Kendrapada having the demand of 75, 100, 100 and 30 units respectively. The transportation cost per unit in rupees is given in the following table:

Godown→ Company↓	Sambalpur	Jagatsinghpur	Jajpur	Kendrapada	Supply
Bhubaneswar	90	90	100	100	200
Cuttack	50	70	130	85	100
Demand	75	100	100	30	

Make a production plan to minimize the total cost of transportation.

Solution :

Here, the total demand is greater than the total supply by 305-300=5 units so this transportation problem is unbalanced. Now add a dummy company with its monthly capacity 5 units having zero transportation cost as follows.

Godown→ Company↓	Sambalpur	Jagatsinghpur	Jajpur	Kendrapada	Supply
Bhubaneswar	90	90	100	100	200
Cuttack	50	70	130	85	100
Dummy	0	0	0	0	5
Demand	75	100	100	30	105

Then by using VAM we can obtain an initial basic feasible solution. The penalty table for row and column can be constructed as follows:

Destination→ Source↓	Sambalpur	Jagatsinghp ur	Jajpur	Kendrapada	Supply	P_I	P_{II}	P_{III}	P_{IV}
Bhubaneswar	90	90 (75)	100 (95)	100 (30)	~~200~~ ~~105~~ ~~30~~	0	0	10	10

					0				
Cuttack	50 (75)	70 (25)	130	85	~~100~~ ~~25~~ 0	20	20	15	15
Dummy	0	0	0 (5)	0	~~5~~ 0	0	-	-	-
Demand	~~75~~ 0	~~100~~ ~~75~~ 0	~~100~~ ~~95~~ 0	30	105				
P_I	50	70	100↑	85					
P_{II}	40↑	20	30	15					
P_{III}	-	20	30↑	15					
P_{IV}	-	20↑	-	15					

In the above table the number of independent non-negative allocated cells is 6 which is equal to m+n-1 = 3+4-1 = 6. So, here we can get a non-degenerate basic feasible solution. The initial basic feasible solution = 90×75+100×95+100×30+50×75+70×25= 22050 rupees.

Now, we can apply MODI method to determine the optimum solution. We determine a set of numbers u_i and v_j for each row and column such that $u_i + v_j = c_{ij}$ for each occupied cell. By starting we give $u_1 = 0$ as the 1st row has maximum number of allocations.

$$C_{12} = u_1 + v_2 = 90 = 0 + v_2 = 90 \Rightarrow v_2 = 90$$

$$C_{13} = u_1 + v_3 = 100 = 0 + v_3 = 100 \Rightarrow v_3 = 100$$

$$C_{14} = u_1 + v_4 = 100 = 0 + v_4 = 100 \Rightarrow v_4 = 100$$

$$C_{22} = u_2 + v_2 = 70 = u_2 + 90 = 70 \Rightarrow u_2 = -20$$

$$C_{21} = u_2 + v_1 = 50 = -20 + v_1 = 50 \Rightarrow v_1 = 70$$

$$C_{33} = u_3 + v_3 = 0 = u_3 + 100 = 0 \Rightarrow u_3 = -100$$

Then, we find the sum of u_i and v_j for all empty cells and keep it at the bottom left corner of the respective cell. In the next step, we obtain the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for every unoccupied cell and keep it at the bottom right corner of the respective cell.

Destination→ Source↓	Sambalpur	Jagatsinghpur	Jajpur	Kendrapada	u_i
Bhubaneswar	90 70 20	90 (75)	100 (95)	100 (30)	0

Cuttack	50 (75)	70 (25)	130 80 50	85 80 5	-20
Dummy	0 -30 30	0 -10 10	0 (5)	0 0 0	-100
v_j	70	90	100	100	

In the above table some of the $\Delta_{ij} < 0$ (negative) the solution is not optimum. Now we form a fresh basic feasible solution giving maximum allocations to the cell where Δ_{ij} is minimum and negative by taking an occupied cell as empty.

Now, let us operate for an alternative optimal solution. Hence, doing an inspection of the opportunity cost $(z_{ij} - c_{ij})$ of the unoccupied cells in the above table reveals $z_{34} - c_{34} = 0$; which indicates the existence of an alternative optimum solution. Therefore, allocate an unknown quantity θ to the unoccupied cell (3, 4) and thus the modified transportation matrix becomes:

90	90 (75)	100 (95) + θ	100 (30) – θ
50 (75)	70 (25)	130	85
0	0	0 (5) – θ	0 + θ

Here,

Destination→ Source↓	Sambalpur	Jagatsinghpur	Jajpur	Kendrapada	u_i
Bhubaneswar	90 [-20]	90 (75)	100 (100)	100 (25)	0
Cuttack	50 (75)	70 (25)	130 [-50]	85 [-5]	-20
Dummy	0 [-30]	0 [-10]	0 [0]	0 (5)	-100
v_j	70	90	100	100	

maximum quantity that can be allocated to the cell (3, 4) is 5. After putting θ = 5, the alternative solution to the above table can be obtained. From the above table we observe that if the total transportation cost remains same the alternative optimum allocation becomes:

$x_{12} = 75, x_{13} = 100, x_{14} = 25, x_{21} = 75 \text{ and } x_{22} = 25$

1.12 Summary

In this unit we have discusseddifferent methods to solve a transportation problem for minimizing the cost, time and distance. In addition to this, modified distribution method has also been discussed to find the optimum solution of the initial basic feasible solution. In the last section some special cases are alsothoroughly discussed.

1.13 Key Terms

- Transportation problem (TP)
- Feasible Solution (FS)
- Basic feasible solution (BFS)
- Optimal solution (OS)
- Non-degenerate basic feasible solution (NDBFS)
- N.W. Corners Rule (NWC)
- Least cost method (LCM)
- Vogel's Approximation method (VAM)
- Modified Distribution method (MODI) etc.

1.14 Check Your Progress

1) A Transportation problem deals with which of the following situations?
   a. Single product from several sources to a destination
   b. Multi-product from several sources to several destinations
   c. Single product from several sources to several destinations
   d. Single product from a source to several destinations
2) A transportation problem is said to be balanced, in a situation where;
   a. Total demand and total supply are equal
   b. Total source and total destination are equal
   c. Both a and b
   d. None of the above
3) In which of the following conditions, the initial solution of a transportation problem can be obtained?
   a. Solution must be optimum
   b. Solution must be non-degenerate
   c. Rim conditions are satisfied
   d. All of the above
4) The solution of a transportation problem (with m sources and n destinations) is said to be optimum if the number of allocations are ;

a. m+n
b. m+n+1
c. m+n-1
d. (m+1)-(n+1)

5) The initial solution of a transportation problem is obtained by;
a. NWCM
b. LCM
c. VAM
d. All of the above

6) If we want to use opportunity cost value for the non-basic cell to test optimality it must be:
a. Negative number
b. Positive number
c. Most positive number
d. Most negative number

7) Which of the following statements is incorrect?
a. A maximization case is converted into minimization in transportation problem by subtracting every value of the given cost matrix from the maximum value.
b. If in a transportation problem each cost is decreased or increased by a constant amount then the its optimum solution is unaffected.
c. In an optimum solution of a transportation problem,u_i and $v_j$represents the optimum values of a dual problem.
d. Multiple optimum solutions are obtained if there are multiple zeros are for u_i and v_j.

8) In which of the following conditions, a transportation problem has unique and optimum solution?
a. $\Delta_{ij} \leq 0$
b. $\Delta_{ij} < 0$
c. $\Delta_{ij} > 0$
d. $\Delta_{ij} \geq 0$

9) Which among the following expressions can be used to evaluate the value of Δ_{ij}?
a. $c_{ij} - (u_i + v_j)$
b. $c_{ij} - (u_i - v_j)$
c. $c_{ij} + (u_i - v_j)$
d. $c_{ij} + (u_i + v_j$

10) The initial basic feasible solution obtained by which of the following methods is very close to the optimum solution?
a. NWCM
b. LCM

c. VAM
d. None of these above

Possible Answers

1) c 2) a 3) c 4) c 5) d 6) d 7) d 8) c 9) b 10) c

1.15 Further Reading

- Kalavathy, S (2013): Operations Research, 4th Edition, Vikas Publishing House, Pvt. Ltd.
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1.16 Model Questions

a) What is a transportation problem? Explain in detail.

b) Describe an unbalanced transportation problem. When can we say that a transportation problem is balanced? What are its applications?

c) Briefly explain the following methods to find the initial basic feasible solution of a transportation problem :

i) NWCM

ii) LCM

iii) VAM

d) **NWCM**

Raw materials for manufacturing a car are to be transported from origins 1,2,3 to destinations 1,2,3 and 4. The supply at the origins and the demand at the destination is as shown in the table below. Obtain an initial basic feasible solution to the given transportation problem using North west corner method ensuring that the cost of transportation is minimum,

Origin/Destination	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Required	6	10	15	4	35

e) **LCM or MMM**

An iron and steel company has 3 open furnaces and 4 rolling mills. Transportation cost(per quintal) for transporting steel from furnaces to mills is as shown in the table given below.Determine an initial basic feasible solution to the following transportation problem using Least cost method so as to obtain the optimum schedule.

Origin/Destination	D_1	D_2	D_3	D_4	Supply
O_1	2	3	4	6	5
O_2	5	4	6	1	14
O_3	7	9	8	2	16
Required	4	10	6	15	35

f) Briefly explain the conditions which are to be satisfied before proceeding for optimality test in a transportation problem.

g) After obtaining the initial basic feasible solution in a transportation problem, discuss how can we obtain the optimum solution.

h) **VAM**

Find the initial basic solution of the following transportation problem using Vogel's Approximation Method.

Factory/Destination	D_1	D_2	D_3	D_4	Supply
F_1	3	3	4	1	100
F_2	4	2	4	2	125
F_3	1	5	3	2	75
Demand	120	80	75	25	300

i) **(MODI Method)**

Find out the optimal solution of the given transportation problem :

Origin/Destination	D_1	D_2	D_3	D_4	Supply

O_1	2	2	2	1	3
O_2	10	8	5	4	7
O_3	7	6	6	8	5
Demand	4	3	4	4	15

j) **(MODI Method)**

Evaluate an optimal solution of the given transportation problem:

Origin/Destination	A	B	C	Supply
P	2	2	3	10
Q	4	1	2	15
R	1	3	1	40
Demand	20	15	30	65

k) **(MODI Method)**

Solve the following transportation problem for its initial basic solution and optimal solution:

Origin/Destination	A	B	C	D	Supply
P	1	5	3	3	34
Q	3	3	1	2	15
R	0	2	2	3	12
S	2	7	2	4	19
Demand	21	25	17	17	80

l) Prove that the necessary and sufficient condition for existence of a feasible solution to a transportation matrix with m rows and n columns is;

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

where; $a_i$is the availability at i^{th} origin and $b_j$is the requirement at j^{th} destination.

Unit-2
Elementary Assignment Problem

Structure :

2.1 Learning Objectives

The main objective of this unit is to provide elementary knowledge on the concept and application of assignment problems. After reading this unit the reader able to answer the following questions ;

- What is an assignment problem?
- How to tackle an assignment problem?
- How to solve an assignment problem?
- How to tackle some special cases of assignment problems?

2.2 Introductions

An Assignment problem is quite similar to a transportation problem,with the exception thatits objective is tominimize the cost or time ofmanufacturing of the products by allocating one job to one machine or one machine to one job (i.e., one destination to one origin or one origin to one destination only) unlike to minimize cost of transportation like in a transportation problem. In other words,an assignment problem is a special case of a transportation problem where the objective is to assign the number of resources (workers) to the equal number of activities (jobs) at a minimum cost or maximum profit.

2.3 Definition of Assignment Problem

Suppose there are n jobs to be performed and n persons are available for doing these jobs.Assume that every person can do each job at a time though with varying degrees of efficiency.Let C_{ij} be the cost if the i^{th} person is assigned to the j^{th} job. The problem is to find an assignment (i.e., which job should be assigned to which person, on a one-to-one basis) so that the total cost of the performing all the jobs is minimum. The problem of the above kind is known as an assignment problem.

2.4 Assignment Cost Matrix

The above assignment problem can be shown in the form of a n×n cost matrix as follows :

	Jobs							
Persons		1	2	3	…	j	…	n
	1	C_{11}	C_{12}	C_{13}	…	C_{1j}	…	C_{1n}
	2	C_{21}	C_{22}	C_{23}	…	C_{2j}	…	C_{2n}
	3	C_{31}	C_{32}	C_{33}	…	C_{3j}	…	C_{3n}
	…	…	…	…	…	…	…	…
	i	C_{i1}	C_{i2}	C_{i3}	…	C_{ij}	…	C_{in}
	…	…	…	…	…	…	…	…
	n	C_{n1}	C_{n2}	C_{n3}	…	C_{nj}	…	C_{nn}

Where, C_{ij} is the cost if the i^{th} person is assigned to the j^{th} job.

2.5 Mathematical Formulation of Assignment Problem

The assignment problems can be formulated mathematically as follows:

$$Minimize\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Where; i=1, 2, 3,…,n and j = 1, 2, 3, …., n

Subject to the constraints,

$\sum_{i=1}^{n} x_{ij} = 1$ (One job done by i^{th} person) and

$\sum_{j=1}^{n} x_{ij} = 1$ (one person assigned for the j^{th} job)

and $x_{ij} = \begin{cases} 1;\ if\ ith\ person\ is\ assigned\ to\ jth\ job \\ 0;\ otherwise \end{cases}$

Non negative conditions,

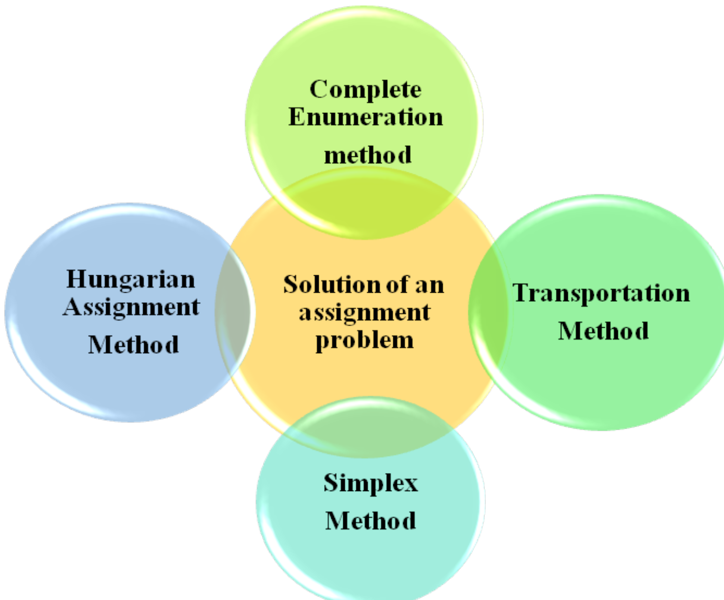
$x_{ij} \geq 0$ for all i and j.

2.6 Difference Between Transportation and Assignment Problem

Transportation Problem	Assignment problem
The number of sources and designations are not necessarily equal.Therefore, the cost matrix may not be a square matrix.	Assignment is done on one-to-one basis so the number of sources is equal to the number of destinations. Therefore, the cost matrixshould be a square matrix.
Here x_{ij}represents the quantity transported fromorigin i to the destination j and it can take any possible non-negative value by satisfying the rim requirements.	Here x_{ij}represents the person i assigned to job jwhich takesvalues either0 or 1 .
In this case the capacity is $a_i$and the requirement id b_j for the source i and destination j.	In this case the capacity and the requirement value is exactly one for each source andeach destination.
A transportation problem is said to be balanced if total supply and total demand are equal.	An assignment problem is said to be balanced if number of rows and columns are equal in a cost matrix.
In this case the problem the problem is unbalanced if the total supply and total demand aredifferent.	In this case the problem is unbalanced if number of rows and columns are not equal in a cost matrix.

2.7 Methods of Solving an Assignment Problem

There are several methods can be used to solve an assignment problem, but the followingfour methods are generally used :



In this section we will discuss only Hungarian assignment method. However, reader can take help from the references mentioned at the end of the unit for the clarification on other methods.

2.8 Hungarian Assignment Method

An effective method for solving an assignment problem is Hungarian assignment method (Reduced matrix method). The solution of the assignment problem using Hungarian method involves in the following steps:

Step-1
- Make a cost table or cost matrix.

Step-2
- If the number of rows is not equal to the number of columns then add the required dummy row or column with zero cost, otherwise proceed to the next step.

Step-3
- Identify the smallest element from each row and subtract it from all the elements from respective rows.

Step-4
- Then identify the smallest element from each column and subtract it from all the elements from respective columns to obtain a modified matrix.

Step-5
- Now, trace as many as possible minimum horizontal and vertical lines to cover all the zeroes in the modified matrix. Denote the total lines to be N and n to be the order of the modified square matrix.

Step-6
- If N = n the the optimal solution of the assignment problem can be obtained and we can make optimal assignment to get desired solution for which go to step-8.

Step-7
- If N < n , then identify the smallest element not covered by previous N lines. Then, subtract it from all other uncovered elements and add it to the intersection point made by horizontal and vertical lines. Then repeat from step-3 until reach step 6.

Step-8
- Then, examine all the rows and circle (○) the single zero found among them to mark it assigned and put cross mark (×) to all the other zeroes lying in the respective column so they can't be taken for future assignment. Repeat same procedure for columns.

Step-9
- Then, repeat from step-3 until any of the following situation arises:
  i) If there is no unmarked zero left, then go to the step-10
  ii) If there is any unmarked zero then circle (○) any one arbitarily and cross (×) the remaings in its row or column. Repeat the same procedure untill step-9 (i) satisfied.

Step-10
- Finally, one circled zero can be identified in each row and each column. Now, the assignment corresponding to these circled zeroes will give us the optimal assignment.

Example : 1

Obtain i) the optimal job assignment and ii) the minimum cost of assignments for the following cost table.

		Jobs				
		1	2	3	4	5
Mechanics	A	10	3	3	2	8
	B	9	7	8	2	7
	C	7	5	6	2	4
	D	3	5	8	2	4
	E	9	10	9	6	10

Solution :

Following the above mentioned steps , we start with identifying the smallest element from each row and subtracting it from each element in that respective row; we get,

	1	2	3	4	5
A	8	1	1	0	6
B	7	5	6	0	5
C	5	3	4	0	2
D	1	3	6	0	2
E	3	4	3	0	4

Then, we identify the smallest element from each column and subtract it from each element in that respective column; we get,

	1	2	3	4	5
A	7	0	0	0	4
B	6	4	5	0	3
C	4	2	3	0	0
D	0	2	5	0	0
E	2	3	2	0	2

Now, after tracing minimum number of horizontal and vertical lines covering all the zeroes we get ;

	1	2	3	4	5
A	7	0	0	0	4
B	6	4	5	0	3
C	4	2	3	0	0
D	0	2	5	0	0
E	2	3	2	0	2

Here, the number of lines drawn to cover all zeroes is N=4. The order of the above matrix is; n=5. As, N< n wemove for the next operation, by subtracting the smallest uncovered elements from the rest uncovered elements and adding it to the intersection elements in the cost table. Then again making least possible horizontal and vertical lines to cover all zeroes.

	1	2	3	4	5
A	9	0	0	2	6
B	6	2	3	0	3
C	4	0	1	0	0
D	0	0	3	0	0
E	2	1	0	0	2

Now, as N=5 and n=5 i.e., N=n, we can get the optimal solution of this assignment.

Examining all the rows and columns then assigning circle (○) to the single zeroes and cross (×) to remaining zeroes lying in the respective rows and columns we get the following cost table;

	1	2	3	4	5
A	9	(0)	0×	2	6
B	6	2	3	(0)	3
C	4	0×	1	0×	(0)
D	(0)	0×	3	0×	0×
E	2	1	(0)	0×	2

Asrow A has two zeroes, we can proceed to row B. Row B has only one zero in the cell(B, 4). So, we can assign it by making a circle (○) in it and cross (×) rest of the zeroes in the column 4 having cell (C, 4), (D, 4) and (E, 4).Again, as row C and D has more than one zero, we canproceed to rowE (where one was zero crossed in the earlier assignment).As the row has one zero available for assignment, we can assign it with the cell (E, 3) by making a circle (○) and cross (×) rest of the zeroes in the row E and column 3 having the cell (E, 4) and (A, 3).

Checking column wise, column 1 has asingle zero in the cell (D, 1) we assign it by a circle (○) and cross (×) rest of the zeroes in row D having cells (D, 2), (D, 4) and (D, 5).Column 2 has two zeroes and column 3, 4 has no zeroes left to assign so move to the column 5. In column 5 only one zero is left, sowe assign it with circle (○) in the cell (C, 5) and cross (×) the zero in the cell (C, 2). Finally, only one zero is left in cell (A, 2) as unassigned so we assign it by making a circle (○) in it.Thus, we have reached the final assignments as follows;

		Jobs				
		1	2	3	4	5
Mechanics	A		**3**			
	B				**2**	
	C					**4**
	D	**3**				
	E			**9**		

Now, the final job assignments to the different mechanics are, 1→D, 2→A, 3→E, 4→B and 5→C with the minimum cost 3+3+9+2+4= 21 rupees.

2.9 Some Special Cases

There are some special cases involved in an assignment problem and they are:

i) Multiple solutions

ii) Maximization case

iii) Unbiased case

iv) Restrictions on assignment

Let us discuss these cases one by one.

i) Multiple / Alternative solutions

There are several situations where in the final assignment table no rows and columns have single zero to assign the job. In that situation we may have different alternative or multiple solutions, which can be assigned arbitrarily. See the following example for better understanding.

Example-2 (Multiple / Alternative solutions)

There are four different workers available for different works. The following cost matrix shows the cost of doing work i (i= 1, 2, 3 and 4) by the worker j (= 1, 2, 3 and 4).

worker→ works↓	J1	J2	J3	J4
I1	5	7	11	6
I2	8	5	9	6
I3	4	7	10	7
I4	10	4	8	3

How the different works be assigned to various workers so that the total cost can be minimized?

Solution :

First, we identify the smallest element from each row (and column) and subtract this from each element from the respective rows (and columns) , then get the following reduced matrix;

worker→ works↓	J1	J2	J3	J4
I1	0	2	2	1
I2	3	0	0	1
I3	0	3	2	3
I4	7	1	1	0

Now draw the minimum number of horizontal and vertical lines to cover all the zeros as follows;

worker→ works↓	J1	J2	J3	J4
I1	0	2	2	1
I2	3	0	0	1
I3	0	3	2	3
I4	7	1	1	0

Here, the minimum number of lines to cover all zero is N=3, which is less than the order of given matrix i.e., n=4. So, we form a modified matrix, by subtracting the smallest uncovered cost from all the uncovered cost and adding to the cost which is at the point of intersection of traced lines.

Now the modified matrix becomes;

worker→ works↓	J1	J2	J3	J4
I1	0	1	1	1
I2	4	0	0	2
I3	0	2	1	3
I4	7	0	0	0

Again, the total of lines used to cover all zeros (N=3) not equal to the order (n=4) of the given matrix. So, proceeding in similar manner we get another modified matrix as follows;

worker→ works↓	J1	J2	J3	J4
I1	0	0	0	0
I2	5	0	0	2
I3	0	1	0	2
I4	8	0	0	0

Now, N=4 and n=4 i.e., N=n, thus we can get the optimal solution of this assignment problem. So, examining all the rows and columns then assigning circle (○) to the single zeros and cross (×) to remaining zeros lying in the respective rows and columns we get following two modified cost matrices as follows;

Cost matrix-1

worker→ works↓	J1	J2	J3	J4
I1	0×	0×	0×	(0)
I2	5	(0)	0×	2
I3	(0)	1	0×	2
I4	8	0×	(0)	0×

Cost matrix-2

worker→ works↓	J1	J2	J3	J4
I1	(0)	0×	0×	0×
I2	5	(0)	0×	2
I3	0×	1	(0)	2
I4	8	0×	0×	(0)

Since no row and column have single zero, we can have multiple assignments and thus multiple solutions as follows;

Cost matrix-1

worker→ works↓	J1	J2	J3	J4
I1				**6**
I2		**5**		
I3	**4**			
I4			**8**	

Cost matrix-2

worker→ works↓	J1	J2	J3	J4
I1	**5**			
I2		**5**		
I3			**10**	
I4				**3**

Thus, the final work assignments to the different workers are,

case-1: I1→J4, I2→J2, I3→J1 and I4→J3 with the minimum cost 6+5+4+8= 23 rupees.

case-2: I1→J1, I2→J2, I3→J3 and I4→J4 with the minimum cost 5+5+10+3= 23 rupees.

ii) Maximization / Alternative solutions

This case particularly deals with maximizing total profit. To handle this type of assignment problemwe first need to convert the given profit matrix into the loss matrix by identifying the highest value in that matrix and subtracting every element from this highest profit. Then the further procedure is as usual to get desired optimum assignment.

Example : 3 (Maximization case)

In a tailor shop there are 4 tailoring machines and 5 jobs are available. The expected profit for each machine for each job is given in the form of a profit matrix as follows.

		Jobs				
		J1	J2	J3	J4	J5
Machines	M1	62	78	50	111	82
	M2	71	84	61	73	59
	M3	87	92	111	71	81
	M4	48	64	87	77	80

By using an assignment approach, find out the assignments ofdifferent jobs to different machines which will maximize the profit. Which job should be declined?

Solution :

In this case the profit matrix doesn't have same number of rows and columns. To make a square profit matrix we put a dummy Machine M5 with all the profit value as '0'.

Profit Matrix	J1	J2	J3	J4	J5
M1	62	78	50	111	82
M2	71	84	61	73	59
M3	87	92	111	71	81
M4	48	64	87	77	80
M5	0	0	0	0	0

Then, convert the above profit matrix as a loss matrix by doing subtraction of all the elements from the highest profit value i.e., 111.

Loss Matrix	J1	J2	J3	J4	J5
M1	49	33	61	0	29
M2	40	27	50	38	52
M3	24	19	0	40	30
M4	63	47	24	34	31
M5	111	111	111	111	111

Now the above cost matrix (loss matrix) can be operated as usual way to obtained best assignment solution. So, subtracting the smallest element from all the remaining elements in each row and column weget the modified matrix as follows.

	J1	J2	J3	J4	J5
M1	49	33	61	0	29
M2	13	0	23	11	25
M3	24	19	0	40	30
M4	39	23	0	10	7
M5	0	0	0	0	0

Now we draw the minimum number of lines to cover all the zeroes as follows:

	J1	J2	J3	J4	J5
M1	49	33	61	0	29
M2	13	0	23	11	25
M3	24	19	0	40	30
M4	39	23	0	10	7
M5	0	0	0	0	0

Here, the total number of lines covering all zeroes is N=4, which is less than the order of the cost matrix n=5. So, again we identify the smallest element from uncovered elements, then subtract it from remaining uncovered elements and add it to the intersection elements. Then a modified matrix can be obtained by making least possible horizontal and vertical lines to cover all zeroes as follows.

	J1	J2	J3	J4	J5
M1	49	40	68	0	29
M2	6	0	23	5	18
M3	17	19	0	33	23
M4	32	23	0	3	0
M5	0	7	7	0	0

Now, N=5 and n=5 i.e., N=n, we can get the optimal solution of this assignment problem.

Examining all the rows and columns then assigning circle (○) to the single zeroes and cross (×) to remaining zeroes lying in the respective rows and columns we get amodified cost table as follows;

	J1	J2	J3	J4	J5
M1	49	40	68	(0)	29
M2	6	(0)	23	5	18
M3	17	19	(0)	33	23
M4	32	23	0×	3	(0)
M5	(0)	7	7	0×	0×

Thus, we reachat the optimum allocations highlighted as in the table below:

		Jobs				
		J1	J2	J3	J4	J5
Machines	M1				**111**	
	M2		**84**			
	M3			**111**		
	M4					**81**
	M5	**48**				

As M5 is a dummy machine the job assigned for it should be declined. Thus, the final work assignments to the different machines are, J1→M5 (declined), J2→M2, J3→M3, J4→M1 and J5→M4 with the minimum cost 84+111+111+81= 387 rupees.

iii) Unbalanced case

An assignment problem is said to be un-balanced if the number of rows is not equal to the number of columns of a cost matrix. To handle with this kind of situation we can add the required dummy rows and dummy columns having entries zero (0). Here is an illustration in the example below.

Example : 4 (Unbalanced case)

There are 5 workers to be assigned to do 4 works in such a way that, one worker can be assigned to do only a single work. The total time required in hours to complete a work by a worker is given in the following table;

		Workers				
		I	II	III	IV	V
Works	A	4	3	6	2	7
	B	10	12	11	14	16
	C	4	3	2	1	5
	D	8	7	6	9	6

Solve the above assignment problem so as to minimize total time required to complete the whole work by differentmachines and also find out which worker is unassigned.

Solution:

As the given cost matrix doesn't have equal number of rows and columns it is known as an unbalanced problem. To make it balancedwe add here a dummy work 'E' with all the entries '0'. Now the modified matrix becomes;

	I	II	III	IV	V
A	4	3	6	2	7
B	10	12	11	14	16
C	4	3	2	1	5
D	8	7	6	9	6
E	0	0	0	0	0

In the next step, identifying the smallest element from each row and subtracting this from each element from the respective rows we get;

	I	II	III	IV	V
A	2	1	4	0	5
B	0	2	1	4	6
C	3	2	1	0	4
D	2	1	0	3	0
E	0	0	0	0	0

Since all the columns have their lowest element zero, no need to do the above operation for the columns separately. Now draw the minimum number of lines to cover all the zeroes as follows;

	I	II	III	IV	V
A	2	1	4	0	5
B	0	2	1	4	6
C	3	2	1	0	4
D	2	1	0	3	0
E	0	0	0	0	0

Now, the total number of lines covering all zeroes is N=4, which is less than the order of the cost matrix n=5. So, again identify the smallest element from uncovered elements, then subtract it from remaining other uncovered elements and add with the intersection elements. Then the modified matrix can be obtained by making least possible horizontal and vertical lines to cover all zeroes as follows.

	I	II	III	IV	V
A	2	0	3	0	4
B	0	1	0	4	5
C	3	1	0	0	3
D	3	1	0	4	0
E	1	0	0	1	0

Now, N=5 and n=5 i.e., N=n, we can get the optimal solution of this assignment.

Examining all the rows and columns then assigning circle (○) to the single zeroes and cross (×) to remaining zeroes lying in the respective rows and columns we get the following cost table;

	I	II	III	IV	V
A	2	(0)	3	0×	4
B	(0)	1	0×	4	5
C	3	1	(0)	0×	3
D	3	1	0×	4	(0)
E	1	0×	0×	1	0×

Thus, we reach at the optimum allocation as follows;

		Workers				
		I	II	III	IV	V

Works	A		**3**			
	B	**10**				
	C			**2**		
	D					**6**

Now, the final work assignments to the different workers are, A→II, B→I,C→III and D→V with minimum cost 3+10+2+6= 21 rupees. For worker IV no work has been assigned.

iv) **Restrictions on assignment**

Due to some unavoidable restrictions in an assignment problem, a facility to a particular job is not permitted. To handle such type of situation a very high cost (∞ cost) is assigned to that particular cell and it subsequently exclude the said activity from the optimal solution.

Example : 5 (Restrictions on assignment)

Solve the following assignment problem where two assignments are restricted.

Job→ Facility↓	A	B	C	D	E
P	9	11	15	10	11
Q	12	9	-	10	9
R	-	11	14	11	7
S	14	8	12	7	8

Solution :

In this problem, the matrix is not a square matrix. To make it a square matrix put a dummy facility ‘T’ with all the cost value as ‘0’. In addition to this the facility ‘Q’ can’t be assigned to the job ‘C’ and the facility ‘R’ can’t be assigned to the job ‘A’. So, we assign a very high cost (∞ cost) in the cell (Q, C) and (R, A). Now, we have the modified assignment matrix as;

Job→ Facility↓	A	B	C	D	E
P	9	11	15	10	11
Q	12	9	∞	10	9
R	∞	11	14	11	7
S	14	8	12	7	8
T	0	0	0	0	0

In the next step, identifying the smallest element from each row (and column) and subtracting this from each element from the respective rows (and columns) we get the following reduced matrix;

Job→ Facility↓	A	B	C	D	E
P	0	2	6	1	2
Q	3	0	∞	1	0
R	∞	4	7	4	0
S	7	1	5	0	1
T	0	0	0	0	0

Proceeding the usual manner, we get the following zero assignment matrix;

Job→ Facility↓	A	B	C	D	E
P	(0)	2	6	1	2
Q	3	(0)	∞	1	0✕
R	∞	4	7	4	(0)
S	7	1	5	(0)	1
T	0✕	0✕	(0)	0✕	0✕

Since, in each row and column as element is assigned, we reached at an optimal assignment as follows;

Job→ Facility↓	A	B	C	D	E
P	**9**				
Q		**9**			
R					**9**
S				**11**	
T			**12**		

Now, the final facility assignments to the different jobs are, P→A, Q→B, R→E, S→D and the job 'C' is remains unassigned as 'T' is a dummy facility assigned to it. Thus, the minimum cost for this assignment is = 9+9+9+11= 38 rupees.

2.10 Summary

To solve an assignment problem, we can use several methods, among them Hungarian method is easy to use and simple to carryout. In this unit a detailed method has been explained. However, readers can use other method with their own interest (see references).

2.11 Key Terms

- Assignment cost matrix
- Mathematical formulation
- Transportation problem
- Assignment problem
- Hungarian assignment method
- Special cases of assignment
- Multiple solution
- Maximization case
- Unbalanced case
- Restriction on assignments.

2.12 Check Your Progress

State whether the following statements are True or False.

a. In an assignment problem the jobs are allocated at a maximum cost.

b. The linear programming model for an assignment problem has constraints ; one for jobs and other for persons who will do the job.

c. In a balanced assignment model the cost matrix need not be a square matrix.

d. In an unbalanced assignment problem, dummy row and dummy column has been entered.

e. We can't assign the zeroes in a row or column of a cost matrix arbitrarily at the starting of assignment.

f. An assignment problem is a special case of transportation problem.

g. To solve an assignment problem we can use the reduced matrix method.

h. If we have n persons and n jobs then there are n solutions to this assignment problem.

i. The maximum number of lines covering all the zeroes in a reduced cost matrix is n.

j. An assignment problem can also be solved by simplex method.

Possible Answers

a. F b. T c. F d. T e. T f. T g. F h. F i. T j. T

2.13 Further Reading

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2.14 Model Questions

- A glass manufacturing company has 5 jobs and 5 machines are available for these jobs. The cost involved with respect to different jobs are given in the following cost table.

		Machines				
		M1	M2	M3	M4	M5
Jobs	J1	13	8	16	18	19
	J2	9	15	24	9	12
	J3	12	9	4	4	4
	J4	6	12	10	8	13
	J5	15	17	18	12	20

Solve the above assignment problem so that all the machines will be assigned with different jobs with minimized total cost. (Ans.: Optimum cost= 42 Rupees)

- There are four personsand four jobs are available for them. The following table shows the cost involving theperson with respect to their jobs.

		Persons			
		P1	P2	P3	P4
Jobs	J1	5	7	11	6
	J2	8	5	9	6
	J3	4	7	10	7
	J4	10	4	8	3

Solve the above assignment problem so that different jobs can be assigned to different persons so that the total cost can be minimised. (Ans.: Optimum cost= 23 Rupees)

- A cost table involving different costs for workers assigned with different machines respectively as given below. Solve this problem of assignment so as to minimise the assignment cost.

			Workers			
		W1	W2	W3	W4	W5
Machines	M1	30	25	33	35	36
	M2	23	29	38	23	26
	M3	30	27	22	22	22
	M4	25	31	29	27	32
	M5	27	29	30	24	32

(Ans.: Optimum cost = 122 rupees)

- **Unbalanced Case**

  A toy manufacturing company has 4 machines to make 3 types of toys. Manufacturing of each toy can be assigned to only one machine. The cost matrix for this manufacturing process is given below. Solve this assignment problem to minimise the total cost of manufacturing toys by the particular company.

		Machines			
		M1	M2	M3	M4

Toys	T1	18	24	28	32
	T2	8	13	17	18
	T3	10	15	19	22

(Ans.: Optimum cost= 50 Rupees and M4 assigned no job)

- **Maximization case**

Consider there are 5 salesmen and 5 different sale markets. The profit matrix associated with their business in different sale markets given below.

		Salesmen				
		A	B	C	D	E
Sale Markets	I	32	38	40	28	40
	II	40	24	28	21	36
	III	41	27	33	30	37
	IV	22	38	41	36	36
	V	29	33	40	35	39

Obtain the assignments of the salesmen to the different sale markets so that the profit will be maximized. (Ans.: Maximum profit= 191 rupees)

□□□

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